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BY

BENJAMIN P EIRCE, LL. D.

Perkins Professor of Math. and Astron. at Harvard University
and Superintendent of the United States Coast Survey.

Read before the National Academy of Sciences.

WASHINGTON CITY.

1870

Math 1708.70

1870. July 5.
Gift of
Prof. Genl. Peirce.

To my friends.

This work has been the pleasantest mathematical effort of my life. In no other have I seemed to myself to have received so full a reward for my mental labor in the novelty and breadth of the results. I presume that to the uninitiated the formulae will appear cold and cheerless. But let it be remembered that, like other mathematical formulae, they find their origin in the divine source of all geometry. Whether I shall have the satisfaction of taking part in their exposition, or whether that will remain for some more profound expositor, will be seen in the future.

B. P.

Linear Associative Algebra

1. Mathematics is the science which draws necessary conclusions.

This definition of mathematics is wider than that which is ordinarily given, and by which its range is limited to quantitative research. The ordinary definition like that of other sciences, is objective; whereas this is subjective. Recent investigations, of which quaternions is the most noteworthy instance, make it manifest that the old definition is too restricted. The sphere of mathematics is here extended, in accordance with the derivation of its name to all demonstrative research, so as to include all knowledge strictly capable of dogmatic teaching. Mathematics is not the discoverer of laws, for it is not induction; neither is it the framer of theories, for it is not hypothesis; but it is the judge over both, and it is the arbiter to which either must refer its claims; and neither law can rule nor theory explain without the sanction of mathematics. It deduces from a law all its consequences, and develops them into the suitable form for comparison with observation

Definition of mathematics

3.

and thereby measures the strength of the argument from observation in favor of a proposed law, or of a proposed form of application of a law.

Mathematics, under this definition, belongs to every enquiry, moral as well as physical. Even the rules of logic, by which it is rigidly bound could not be deduced without its aid. The laws of argument admit of simple statement, but they must be curiously transposed, before they can be applied to the living speech and verified by observation. In its pure and simple form the syllogism cannot be directly compared with all experience, or it would not have required an Aristotle to discover it. It must be transmuted into all the possible shapes, in which reasoning loves to clothe itself. The transmutation is the mathematical process in the establishment of the law. Of some sciences, it is so large a portion that they have been quite abandoned to the mathematician which may not have been altogether to the advantage of philosophy. Such is the case with geometry and analytic mechanics. But in many other sciences, as in all those of mental philosophy and most of the branches of natural history, the

deductions are so immediate and of such simple construction, that it is of no practical value to separate the mathematical portion and subject it to isolated discussion.

2. The branches of mathematics are as various as the sciences to which they belong, and each subject of physical enquiry has its appropriate mathematics. In every form of material manifestation, there is a corresponding form of human thought, so that the human mind is as wide in its range of thought as the physical universe in which it thinks. The two are wonderfully matched.

But where there is a great diversity of physical appearance, there is often a close resemblance in the processes of deduction. It is important, therefore, to separate the intellectual work from the external form. Symbols must be adopted which may serve for the embodiment of forms of argument, without being trammelled by the conditions of external representation or special interpretation. The words of common language are usually unfit for this purpose, so that other symbols must be adopted, and mathematics treated by such symbols is called

algebra. Algebra is then formal mathematics.

3. All relations are either qualitative or quantitative. Qualitative relations can be considered by themselves without regard to quantity. The algebra of such enquiries may be called logical algebra, of which a fine example is given by Boole.

Quantitative relations may also be considered by themselves without regard to quality. They belong to arithmetic and the corresponding algebra is the common or arithmetical algebra.

In all other algebras both relations must be combined, and the algebra must conform to the character of the relations.

4. The symbols of an algebra, with the laws of combination constitute its language; the methods of using the symbols in the drawing of inferences is its art; and their interpretation is its scientific application. This three-fold analysis of algebra is adopted from President Hill of Harvard University, and is made the basis of a division into books.

Language of algebra

Book 1.

The Language of Algebra

5. The language of algebra has its alphabet, vocabulary and grammar.

6. The symbols of algebra are of two kinds, one of which represents its fundamental conceptions and may be called its letters, and the other represents the relations or modes of combination of the letters and are called the signs.

7. The alphabet of an algebra consists of its letters; the vocabulary defines its signs and the elementary combinations of its letters; and the grammar gives the rules of composition by which the letters and signs are united into a complete and consistent system.

The Alphabet.

8. Algebras may be distinguished from each other by the number of their independent fundamental conceptions, or of the letters of

their alphabet. Thus an algebra, which has only one letter in its alphabet is a single algebra; one which has two letters is a double algebra, one of three letters, a triple algebra; one of four letters, a quadruple algebra and so on.

This artificial division of the algebras is cold and uninteresting like the artificial Linnean system of botany. But it is useful in a preliminary investigation of algebras, until a sufficient variety is obtained to afford the material for a natural classification.

Each fundamental conception may be called an unit, and thus each unit has its corresponding letter, and the two words, unit and letter, may often be used indiscriminately in place of each other, when it cannot cause confusion.

9. The present investigation not usually extending beyond the sextuple algebra, limits the demand of the algebra for the most part to six letters, and the six letters, i, j, k, l, m and n, will be restricted to this use except in special cases.

Substitution of letters

10. For any given letter another may be substituted provided a new letter represents a combination of the original letters of which the replaced letter is a necessary component.

For example, any combination of two letters, which is entirely dependent for its value upon both of its components, such as their sum, difference or product may be substituted for either of them.

This principle of the substitution of letters is radically important, and is a leading element of originality in the present investigation, and without it such an investigation would have been impossible. It enables the geometer to analyse an algebra, reduce it to its simplest and characteristic forms, and compare it with other algebras. It involves in its principle a corresponding substitution of units of which it is in reality the formal representative.

There is however, no danger in working with the symbols, irrespective of the ideas attached to them, and the consideration of the change of the original conceptions may be safely reserved for the book of interpretation.

11. In making the substitution of letters the original letter will be preserved with the distinction of a subscript number.

Thus for the letter i there may successively be substituted i_1, i_2, i_3 , &c. In the final forms, the subscript numbers can be omitted, and they may be omitted at any period of the investigation, when it will not produce confusion.

It will be practically found that these subscript numbers need scarcely ever be written. They pass through the mind, as a sure ideal protection from erroneous substitution but disappear from the writing with the same facility with which those evanescent chemical compounds, which are essential to the theory of transformation, escape the eye of the observer.

12. A pure algebra is one in which every letter is connected by some indissoluble relation with every other letter.

13. When the letters of an algebra can be separated into two groups, which are mutually independent, it is a mixed algebra. It is mixed even when there are letters common to the two groups, provided those which are not common

Pure and mixed algebras

to the two groups are mutually independent.

Were an algebra employed for the simultaneous discussion of distinct classes of phenomena such as those of sound and light, and were the peculiar units of each class to have their appropriate letters, but were there no recognized dependence of the phenomena upon each other, so that the phenomena of each class might have been submitted to independent research, the one algebra would be actually a mixture of two algebras, one of which was appropriate to sound and the other to light.

It may be farther observed that, when in such a case as this, the component algebras are identical in form, they are reduced to the case of one algebra with two diverse interpretations.

The Vocabulary.

14. Letters which are not appropriated to the alphabet of the algebra, may be used in any convenient sense. But it is well to employ the small letters for expressions of common algebra, and the capital letters for those of the algebra

under discussion.

There must however be exceptions to this notation, thus the letter $\mathcal{D}$ will denote the derivative of an expression to which it is applied and Σ the summation of cognate expressions, and other exceptions will be mentioned as they occur. Greek letters will generally be reserved for angular and functional notation.

15. The three symbols $\mathcal{J}$, $\mathcal{D}$ and $\mathcal{G}$ will be adopted with the signification

$$\mathcal{J} = \sqrt{-1}$$

$\mathcal{D}$ = the ratio of circumference to diameter of circle = 3.1415926536

$\mathcal{G}$ = the base of Napierian logarithms
= 2.7182818285

which gives the mysterious formula

$$\mathcal{J}^2 = \sqrt{\mathcal{G}} = 4.8104.77381$$

16. All the signs of common algebra will be adopted, but any signification will be permitted them which is not inconsistent with their use in common algebra; so that, if by any process an expression, to which they refer is reduced to one of common algebra, they must resume their ordinary signification.

17. The sign $=$ which is called that of equality is used in its ordinary sense to denote that the two expressions which it separates are the same whole, although they represent different combinations of parts.

18. The signs $>$ and $<$ which are those of inequality, and denote "more than" or "less than" in quantity, will be used to denote the relations of a whole to its part, so that the symbol which denotes the part shall be at the vertex of the angle, and that which denotes the whole at its opening. It involves the proposition that the smaller of the quantities is included in the class expressed by the larger. Thus

$$B < A \text{ or } A > B$$

denotes that A is a whole of which B is a part so that all B is A .

If the usual algebra had originated in qualitative instead of quantitative investigations, the use of the symbols might easily have been reversed, for it seems that all conceptions involved in A , must also be involved in B , so that B is more than A in the sense that it involves more ideas.

The combined expression

$$B > C < A$$

denotes that there are quantities expressed by C , which belong to the class A and also to the class B . It implies, therefore, that some B is A and that some A is B . The intermediate C might be omitted, if this were the only proposition intended to be expressed and it might be written

$$B > < A.$$

In like manner the combined expressions

$$B < C > A$$

denotes that there is a class which includes both A and B , which proposition might be written

$$B < > A.$$

19. A vertical mark drawn over either of the preceding signs reverses its signification. Thus

$$A \neq B$$

denotes that B and A are essentially different wholes;

$$A \nsubseteq B \text{ or } B \nsubseteq A$$

denotes that all B is not A , so that if they differ only in quantity, they must bear to each other the relation of

$$A = B \text{ or } A < B.$$

20. The sign $+$ is called plus in common algebra and denotes addition. It may be retained with the same name and the process which it indicates may be called addition. In the simplest cases, it expresses a mere mixture, in which the elements preserve their mutual independence. If the elements cannot be mixed without mutual action and a consequent change of constitution, the mere union is still expressed by the sign of addition, although some other symbol is required to express the character of the mixture as a peculiar compound having properties different from its elements. It is obvious from the simplicity of the union recognized in this sign that the order of the admixture of the elements cannot affect it, so that it may be assumed that

$$A + B = B + A$$

and

$$(A + B) + C = A + (B + C) = A + B + C.$$

21. The sign $-$ is called minus in common algebra, and denotes subtraction. Retaining the same name, the process is to be regarded as the reverse of addition; so that if an expression is first added and then subtracted or the reverse, it disappears

from the result, or in algebraic phrase, it is can-
celed. This gives the equations

$$A + B - B = A - B + B = A$$

and

$$B - B = 0$$

The sign minus is called the negative sign in ordinary algebra, and any term preceded by it may be united with it, and the combination may be called a negative term. This use will be adopted into all the algebras, with the provision that the derivation of the word negative must not transmit its interpretation.

22. The sign $\times$ may be adopted from ordinary algebra with the name of the sign of multiplication but without reference to the meaning of the process. The result of multiplication is to be called the product. The terms which are combined by the sign of multiplication may be called factors; the factor which precedes the sign being distinguished as the multiplier and that which follows it being the multiplicand. The words multiplier, multiplicand and product, may also be conveniently replaced by the terms adopted by Hamilton, of facient, faciend, and factum. Thus the equation

of the product is

multiplier $\times$ multiplicand = product or
 facient $\times$ faciend = factum.

When letters are used, the sign of multiplication can be omitted as in ordinary algebra.

23. When an expression used as a factor in certain combinations gives a product which vanishes, it may be called in those combinations a nil-factor. Where it produces vanishing products as the multiplier, it is nilfacient, but where it is the multiplicand of such a product it is nilfaciend.

24. When an expression used as a factor in certain combinations, overpowers the other factors and is itself the product, it may be called an idemfactor. When in the production of such a result, it is the multiplier, it is idemfacient, but when it is the multiplicand, it is idemfaciend.

25. When an expression which is raised to the square or any higher power, vanishes, it may be called nilpotent; but when raised to a square or higher power, it gives itself as the result it may be called idempotent.

The defining equation of nilpotent and idempotent expressions are respectively $A^n = 0$, and $A^n = A$;

but with reference to idempotent expressions, it will always be assumed that they are of the form

$$\tilde{A} = A,$$

unless it be otherwise distinctly stated.

26. Division is the reverse of multiplication by which its results are verified. It is the process for obtaining one of the factors of a given product when the other factor is given. It is important to distinguish the position of the given factor whether it is facient or faciend. This can be readily indicated by combining the sign of multiplication, and placing it before or after the given factor just as it stands in the product. Thus when the multiplier is the given factor, the correct equation of division is $\text{quotient} = \frac{\text{dividend}}{\text{divisor} \times}$ and the equation of verification is

$$\text{divisor} \times \text{quotient} = \text{dividend}.$$

But when the multiplicand is the given factor, the equation of division is $\text{quotient} = \frac{\text{dividend}}{\times \text{divisor}}$ and the equation of verification is

$$\text{quotient} \times \text{divisor} = \text{dividend}.$$

27. Exponents may be introduced just as in ordinary algebra, and they may even be permitted to assume the forms of the algebra under discussion. There seems to be no necessary restriction to giving them even a wider range and introducing into one algebra the exponents from another! Other signs will be defined when they are needed.

The definition of the fundamental operations is an essential part of the vocabulary, but as it is subject to the rules of grammar which may be adopted, it must be reserved for special investigation in the different algebras.

The Grammar.

28. Quantity enters as a form of thought into every inference. It is always implied in the syllogism. It may not, however, be the direct object of inquiry, so that there may be logical and chemical algebras, into which it only enters accidentally, agreeably to P.3. But where it is recognized, it should be received in its most general form and in all its variety. The algebra is otherwise unnecessarily restricted, and cannot enjoy the benefit of the most fruitful forms of philosophical discussion. But while it is thus introduced as a part of the formal algebra, it is subject to every degree and kind of limitation in this interpretation.

The free introduction of quantity into an algebra does not even involve the reception of

its unit as one of the independent units of the algebra. But without such an unit, no algebra is probably adapted to useful investigation. It is so admitted into quaternions, and its admission seems to have misled some philosophers into the opinion that quaternions was a triple and not a quadruple algebra. This will be the more evident from the form in which quaternions first presents itself in the present investigation and in which the unit of quantity is not distinctly recognizable without a transmutation of the form.⁺

⁺Note. Hamilton's total exclusion of the imaginary of ordinary algebra from the calculus as well as from the interpretation of Quaternions, will not probably be accepted in the future development of this algebra. It evinces the resources of his genius that he was able to accomplish his investigations under these trammels. But like the restrictions of the ancient Geometry, they are inconsistent with the generalizations and broad philosophy of modern science. With the restoration of the ordinary imaginary, quaternions becomes Hamilton's biquaternions. From this point of view, all the algebras of this research would be called bi-algebras.

But with the ordinary imaginary is involved a vast power of research and the distinction of names should correspond, and the algebra which loses it should have its restricted nature indicated by such a name as that of a semi-algebra.

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Quantitative forms

29. The introduction of quantity into an algebra naturally carries with it not only the notation of ordinary algebra, but likewise many of the rules to which it is subject. Thus when a quantity is a factor of a product, it has the same influence whether it be facient or faciend, so that with the notation of § 14, there is the equation

$$A a = a A$$

and in such a product, the quantity a may be called the coefficient.

In like manner, terms which only differ in their coefficients, may be added by adding their coefficients thus

$$(a \pm b) A = a A \pm b A = A a \pm A b = A (a \pm b)$$

20. The exceeding simplicity of the conception of an equation, involves the identity of the equations

$$A = B \text{ and } B = A$$

and the substitution of B for A in every expression so that $MA \pm C = MB \pm C$

or that, the members of an equation may be mutually transposed, or simultaneously increased or decreased or multiplied or divided by equal expressions.

31. How far the principle of § 16 limits the

extent within which the ordinary symbols may be used, cannot easily be decided. But it suggests limitations which may be adopted during the present discussion, and leave an ample field for curious investigation.

The distributive principle of multiplication may be adopted; namely, the principle that the product of an algebraic sum of factors into or by a common factor, is equal to the corresponding algebraic sum of the individual products of the various factors into or by the common factor; and it is expressed by the equations

$$(A \pm B)C = AC \pm BC.$$

$$C(A \pm B) = CA \pm CB.$$

32. The associative principle of multiplication may be adopted, namely that the product of successive multiplications is not affected by the order in which the multiplications are performed, provided there is no change in the relative position of the factors; and it is expressed by the equations

$$ABC = (AB)C = A(BC).$$

This is quite an important limitation, and the algebras which are subject to it will be called associative.

Commutative principle and linear algebra.

33. The principle that the value of a product is not affected by the relative position of the factors is called the commutative principle and is expressed by the equation

$$AB = BA.$$

This principle is not adopted in the present investigation.

34. An algebra in which every expression is reducible to the form of an algebraic sum of terms, each of which consists of a single letter with a quantitative coefficient, is called a linear algebra.⁺ Such are all the algebras of the present investigation.

35. Wherever there is a limited number of independent conceptions, a linear algebra may be adopted. For a combination, which was not reducible to such an algebraic sum as those of linear algebra, would be to that extent independent of the original conceptions, and would be an independent conception additional to those

⁺ Note. In the various algebras of De Morgan's "Triple Algebra" the distributive, associative and commutative principles were all adopted, and they were all linear.

which were assumed to constitute the elements of the algebra.

36. An algebra in which there can be complete interchange of its independent units, without changing the formulae of combination, is a completely symmetrical algebra; and one in which there may be a partial interchange of its units is partially symmetrical. But the term symmetrical should not be applied, unless the interchange is more extensive than that involved in the distributive and commutative principles. An algebra in which the interchange is effected in a certain order which returns into itself is a cyclic algebra.

Thus quaternions is a cyclic algebra, because in either of its fundamental equations such as

$$i^2 = -1$$

$$ij = -ji = k$$

$$ijk = -1$$

there can be an interchange of the letters in the order i, j, k, i , each letter being changed into that which follows it. The double algebra in which

$$i^2 = i, \quad ij = i$$

$$j^2 = j, \quad ji = j$$

is cyclic because the letters are interchangeable in the order i, j, i ; but neither of these algebras is commutative.

37. When an algebra can be reduced to a form in which all the letters are expressed as powers of some one of them, it may be called a potential algebra. If the powers are all squares, it may be called quadratic; if they are cubes, it may be called cubic; and similarly in other cases.

Linear associative algebra.

38. All the expressions of an algebra are distributive, whenever the distributive principle extends to all the letters of the alphabet.

For it is obvious that in the equation

$$(i+j)(k+l) = ik + jk + il + jl$$

each letter can be multiplied by an integer, which gives the form

$$(ai + bj)(ck + dl) = acik + bcjk + adil + bdjl.$$

in which a, b, c , and d are integers. The integers can have the ratios of any four real numbers, so that by simple division, they can be reduced

to such real numbers. Other similar equations can also be formed by writing for a and b , a , and b , or for c and d , c , and d , or by making both these substitutions simultaneously. If then the two first of these new equations are multiplied by 1 and the last by -1 ; the sum of the four equations will be the same as that which would be obtained by substituting for a , b , c and d , $a+1a$, $b+1b$, $c+1c$, and $d+1d$. Hence a , b , c and d may be any numbers, real or imaginary, and in general whatever mixtures A , B , C and D may represent of the original units under the form of an algebraic sum of the letters, i , j , k , &c we shall have

$$(A+B)(C+D) = AC + BC + AD + BD.$$

which is the complete expression of the distributive principle

39. An algebra is associative, whenever the associative principle extends to all the letters of its alphabet.

for if

$$A = \Sigma (a_i) = a_1 i + a_2 j + a_3 k + \dots$$

$$B = \Sigma (b_i) = b_1 i + b_2 j + b_3 k + \dots$$

$$C = \Sigma (c_i) = c_1 i + c_2 j + c_3 k + \dots$$

it is obvious that

Always an idempotent or nilpotent unit

$$AB = \sum (a b, i j)$$

$$BC = \sum (b c, i j)$$

$$(AB)C = \sum (a b, c, i j k) = A(BC) = A B C$$

which is the general expression of the associative principle.

40. In every linear associative algebra, there is at least one idempotent or one nilpotent expression.

Take any combination of letters at will and denote it by A . The square is generally independent of A , and its cube may also be independent of A and A^2 . But there cannot be more powers of A , which are independent of A and of each other, than there are letters of the alphabet; so that there must be some least power of A , which is dependent upon the inferior powers. The mutual dependence of the powers of A , may be expressed in the form of an equation, of which the first member is an algebraic sum, such as

$$\sum_m (a_m A^m) = 0.$$

All the terms of this equation, which involve the square and higher powers of A may be combined and expressed as BA , so that B is itself an algebraic sum of powers of A , and the equation might be written

$$BA + a, A = (B + a,) A = 0.$$

It is easy to deduce from this equation, successively,

$$(B + a,) A^m = 0$$

$$(B + a,) B = 0$$

$$\left(-\frac{B}{a,}\right)^2 = -\frac{B}{a,}$$

so that $-\frac{B}{a,}$ is an idempotent expression. But if $a,$ vanishes, this expression becomes infinite, and instead of it, we have the equation

$$B^2 = 0$$

so that B is a nilpotent expression.

41. When there is an idempotent expression in a linear associative algebra; it can be assumed as one of the independent units, and represented by one of the letters of the alphabet; and it may be called the basis.

The remaining units can be so selected as to be separable into four distinct groups.

With reference to the basis, the units of the first group are idemfactors; those of the second group are idemfaciend and nilfacient; those of the third group are idemfacient and nilfaciend; and those of the fourth group are nilfactors.

First The possibility of the selection of all the

Idempotent basis

remaining units as idemfaciend or nilfaciend is easily established. For if i is the idempotent base its definition gives

$$i^2 = i.$$

The product by the basis of another expression such as A , may be represented by B , so that

$$i A = B,$$

which gives

$$i B = i^2 A = i A = B$$

$$i(A-B) = i A - i B = B - B = 0.$$

whence it appears that B is idemfaciend, and $A-B$ is nilfaciend. In other words A is divided into two parts of which one is idemfaciend and the other is nilfaciend; but, either of these parts may be wanting so as to leave A wholly idemfaciend or wholly nilfaciend.

Secondly. The still farther subdivision of these portions into idemfacient and nilfacient is easily shown to be possible by this same method, with the mere reversal of the relative position of the factors. Hence are obtained The required four groups.

The basis itself may be regarded as belonging to the first group.

42. Any algebraic sum of the letters of a group is an expression which belongs to the same group, and may be called factorially homogeneous.

43. The product of two factorially homogeneous expressions, which does not vanish, is itself factorially homogeneous, and its facient name is the same with that of its facient, while its facient name is the same with that of its facient.

Thus if A and B are, each of them, factorially homogeneous they satisfy the equations

$$i(AB) = (iA)B,$$

$$(AB).i = A(Bi),$$

which shows that the nature of the product as a facient is the same with that of the facient A , and its nature as a facient is the same with that of the facient B .

44. Hence, no product which does not vanish can be commutative unless both its factors belong to the same group.

45. Every product vanishes of which the facient is idemfacient, while the facient is nilfacient; or of which the facient is nilfacient while the facient is idemfacient.

For in either case, this product involves the equation

Character of products

$$A B = (A i) B = A (i B) = 0.$$

46. The combination of the propositions of §§ 43 and 45 is expressed in the following form of a multiplication table. In this table, each factor is expressed by two letters, of which the first denotes its name as a faciend, and the second as a facient. The two letters are d and n, of which d stands for idem and n for nil. The facient is written in the left hand column of the table and the faciend in the upper line. The character of the product, when it does not vanish, is denoted by the combination of letters, or when it must vanish by the zero, which is written upon the same line with the facient and in a column under the faciend.

	dd	dn	nd	nn
dd	dd	dn	0	0
dn	0	0	dd	dn
nd	nd	nn	0	0
nn	0	0	nd	nn

47. It is apparent from the inspection of this table, that every expression which belongs to the second or third groups is nilpotent.

48. It is apparent that all commutative products which do not vanish, are restricted to the first and fourth groups.

49. It is apparent that every continuous product which does not vanish, has the same facient name as its first facient, and the same facient name as its last facient.

50. Since the products of the units of a group remain in the group, they cannot serve as the bonds for uniting different groups, which are the necessary conditions of a pure algebra.

Neither can the first and fourth groups be connected by direct multiplication because the products vanish. The first and fourth groups, therefore, require for their indissoluble union into a pure algebra that there should be units in each of the other two groups.

51. In an algebra which has more than two independent units, all the units except the base cannot belong to the second or to the third group. For in this case, each unit taken with the base would constitute a double algebra, and there could be no bond of connection to prevent their separation into distinct algebras.

52. The units of the fourth group are subject to independent discussion, as if they constituted an algebra of themselves. There must be in this group an idempotent or a nilpotent unit. If there is an idempotent unit, it can be adopted as the basis of this group through which the group can be subdivided into subsidiary groups.

The idempotent unit of the fourth group can even be made the basis of the whole algebra, and the first second and third groups will respectively become the fourth, third and second groups for the new basis.

53. When the first group comprises any units except the basis, there is besides the basis another idempotent expression, or there is a nilpotent expression. By a process similar to that of §40 and a similar argument, it may be shown that for any expression A , which belongs to the first group, there is some least power which can be expressed by means of the basis and the inferior powers in the form of an algebraic sum. This condition may be expressed by the equation

$$\sum_m (a_m A^m) + b i = 0.$$

If then h is determined by the ordinary algebraic

equation

$$\sum_m (a_m h^m) + b = 0$$

and if

$$A_1 = A - h i$$

is substituted for A , an equation is obtained between the powers of A , from which an idempotent expression B or a nilpotent expression can be deduced precisely as in § 40.

54. When there is a second idempotent unit in the first group, the basis can be changed so as to free the first group from this second idempotent unit.

Thus if i is the basis, and if j is the second idempotent unit of the first group, the basis can be changed to

$$i_1 = i - j$$

and with this new basis, j passes from the first to the second group

First. The new basis is idempotent since

$$i_1^2 = (i - j)^2 = i^2 - 2ij + j^2 = i - j = i_1,$$

Secondly. The idempotent unit j passes into the fourth group, since

$$i_1 j = (i - j)j = ij - j^2 = j - j = 0,$$

$$j i_1 = j(i - j) = ji - j^2 = j - j = 0.$$

55. With the preceding change of basis, expressions may pass from idempotent to nilpotent, or from idempotent to nilpotent, but not the reverse.

First. For if A is nilpotent with reference to the original basis, it is also by § 45 nilpotent with reference to the new basis; or if it is idempotent with reference to the original basis, it is idempotent with reference to the new basis.

Secondly. All expressions which are idempotent with reference to the original basis, can by the process of § 41 be separated into two portions with reference to the new basis, one portion of which is idempotent and the other is nilpotent; so that the idempotent portion remains idempotent, and the remainder passes from idempotent to nilpotent. The same process may be applied to the idempotents with similar conclusions.

56. It is evident, then that each group can be reduced so as not to contain more than one idempotent unit, which will be its basis. In the groups which bear to the basis the relations of second and third groups, there are only nilpotent expressions.

57. In a group or an algebra which has no idempotent expression, all the expressions are nilpotent.

Take any expression of this group or algebra and denote it by A . If no power of A vanishes, there must be as shown in § 40, some equation between the powers of A of the form

$$\sum_m a_m A^m = 0$$

in which a , must vanish, or else there would be an idempotent expression as is shown in § 40, which is contrary to the present hypothesis.

If then m_0 denotes the exponent of the least power of A , which enters into this equation, and $m_0 + h$ the exponent of the highest power which occurs in it, the whole number of terms of the equation is, at most, $h+1$. If then the equation is multiplied successively by A and each of its powers as high as that of which the exponent is $(m_0 - 1)h$, this highest exponent denotes the number of new equations which are thus obtained. If moreover

$$B = A^{m_0},$$

the highest power of A introduced into these equations is

$$A^{(m_0 - 1)h + m_0 + h} = A^{m_0(h+1)} = B^{h+1}$$

The whole number of the powers of A contained in the equations is $m_0 k + 1$, and $k + 1$ of these are always integral powers of B ; and there remain $(m_0 - 1)k$ in number which are not integral powers of A . There are, therefore, equations enough to eliminate all the powers of A which are not integral powers of B and leave an equation between the integral powers of B and which will generally include the first power of B . From this equation, an idempotent expression can be obtained by the process of § 40, which is contrary to the hypothesis of the proposition.

It cannot, therefore, be that there is any equation such as that here assumed, and therefore there can be no expression which is not nilpotent.

The few cases of peculiar doubt, can readily be solved as they occur; but they always must involve the possibility of an equation between fewer powers of B than those in the equation of A .

58. When an expression is nilpotent, all its powers which do not vanish, are mutually independent.
Let A be the nilpotent expression, of which the n th power is the highest which does not vanish.

There cannot be any equation between these powers of the form

$$\sum_m a_m A^m = 0.$$

For if m_0 were the exponent of the lowest power of A in this equation, the product of the equation by the $(n-m_0)$ th power of A , reduces it to

$$a_{m_0} A^n = 0, \quad a_{m_0} = 0,$$

that is, the m_0 th power of A disappears from the equation, or there is no least power of A in the equation, or, more definitely there is no such equation.

59. In a group or an algebra which contains no idempotent expression, any nilpotent expression may be selected as the basis; but one is preferable which has the greatest number of powers which do not vanish. All the powers of the basis which do not vanish may be adopted as independent units and represented by the letters of the alphabet.

A nilpotent group or algebra may be said to be of the same order with the number of powers which do not vanish of its basis, provided the basis is selected by the preceding principle. Thus if the squares of all its expressions vanish, it is of the first order; if the cubes all vanish

Nilpotent algebras

and not all the squares, it is of the second order, and so on.

60. It is obvious that in a nilpotent group, which is of the same order with the number of letters which it contains, all the letters except the basis are the successive powers of the basis.

61. In a nilpotent group, every expression, such as A , has some least power, which is nilfacient with reference to any other expression, such as B , and which corresponds to what may be called the facient order of B relatively to A ; and in the same way, there is some least power of A which is nilfaciend with reference to B , and which corresponds to the faciend order of B relatively to A . When the facient and faciend orders are treated of irrespective of any especial reference, they must be referred to the base.

The facient order of a product which does not vanish, is not higher than that of its facient; and the faciend order is not higher than that of its faciend.

62. After the selection of the basis of a nilpotent group, among the expressions which are independent of the basis some one may be selected by the same

method, by which the basis was itself selected, which as well as all its powers that are independent of the basis may be adopted as new letters; and again from the independent expressions which remain, new letters may be selected by the same process, and so on until the alphabet is completed. In making these selections regard should be had to the factorial orders of the products.

63. In every nilpotent group, the facient order of any letter which is independent of the basis can be assumed to be as low as the number of letters which are independent of the basis.

Thus if the number of letters which are independent of the basis is denoted by n' and if n is the order of the group, and for the present purpose it is sufficient to regard n' as being less than n , it is evident that any expression A and all its successive products by the powers of the basis (i), as high as the n' th cannot be independent of each other and of the powers of the basis which do not vanish, so that there must be an equation of the form

$$\sum_{m=1}^n a_m i^m + \sum_{m=0}^{n'} b_m i^m A = 0$$

so that it is easy to see that there is always a value

of A , of the form

$$A_1 = A - \sum_{i=1}^n h_i i^m$$

which will give

$$i^m A_1 = 0$$

which corresponds to the condition of this section.

There is a similar condition, which holds in every selection of a new letter by the method of the preceding section.

64. In a nilpotent group, of which the order is less by unity than the number of letters, the letter which is independent of the basis and its powers, may be selected, so that its product into the basis, shall be equal to the highest power of the basis which does not vanish, and its square shall either vanish or shall also be equal to the highest power of the basis which does not vanish. Thus if the basis is i and if the order of the algebra is n , and if j is the remaining letter it is obvious from § 63 that j might have been assumed such that

$$i j = 0$$

which gives

$$i j i = i j^2 = 0$$

and therefore,

Nilpotent algebras

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$$\begin{aligned} j i &= a i^n + b j \\ j^2 &= a' i^n + b' j \\ 0 = j i^{n+1} &= b j i^n = b^n j i = b \\ j i &= a i^n \\ j^2 i &= a j i^n = 0 = b' j^2 = b' \\ j^2 &= a' i^n \end{aligned}$$

so that if $k = \left(\frac{a'}{a^2}\right)^{\frac{1}{n-2}}, \quad k = \left(\frac{a'^{n-1}}{a^n}\right)^{\frac{1}{n-2}}$

$$j_1 = \frac{j}{k}, \quad i_1 = \frac{i}{k}$$

we have

$$j_1 i_1 = i_1^n = j_1^2$$

and i_1 and j_1 can be substituted for i and j , which conforms to the proposition enunciated.

It must be observed, however, that the analysis needs correction when the group is of the second order.

65. In a nilpotent group of the first order, the sign of a product is merely reversed by changing the order of its factors. Thus if

$$A^2 = B^2 = (A+B)^2 = 0,$$

it follows by development, that

$$(A+B)^2 = A^2 + AB + BA + B^2 = AB + BA = 0$$

$$BA = -AB,$$

which is the proposition enunciated.

66. In general, in any nilpotent group of the n th order if (A^s, B^t) denotes the sum of all possible products of the form

$A^{p_1} B^{q_1} A^{p_2} B^{q_2} A^{p_3} B^{q_3} \dots$ in which

$$s = \sum p_i, \quad t = \sum q_i$$

and if

$$s + t = n + 1$$

it will be found that

$$(A^s, B^t) = 0.$$

For since

$$(A + xB)^{n+1} = 0$$

whatever be the value of x , each term multiplied by x must vanish, which gives the proposed equation $(A^s, B^t) = 0$

67. In the first group of an algebra, which has an idempotent basis, all the expressions except the basis may be assumed to be nilpotent. For by the same argument as that of § 53 any equation between an expression, and its successive powers and the basis must involve an equation between another expression which is easily defined and its successive powers without including the basis. But it follows, from the argument of § 57 that such an equation indicates a corresponding

idempotent expression, whereas it is here assumed that, in accordance with § 56, each group has been brought to a form which does not contain any other idempotent expression than the basis. It must be, therefore, that all the other expressions are nilpotent.

68. All the products of the expressions in the first group of an algebra which has an idempotent basis, are independent of the basis.

Thus in the equation

$$AB = -xi + C$$

in which AB and C may be nilpotents of the orders respectively m , n and p . Whence we find

$$0 = A^{m+1}B = -x A^m + A^m C$$

$$A^m C = x A^m$$

$$0 = A^m C^{p+1} = x A^m C^p = x^2 A^m C^{p-1} = \dots = x^{p+1} A^m = x$$

that is, the term $-xi$ vanishes from the product AB .

69. It follows, from the preceding section, that if the idempotent basis were taken away from the first group of which it is the basis, the remaining letters of the first group constitute by themselves a nilpotent algebra.

Conversely. Any nilpotent algebra may be constituted into an algebra with an idempotent basis, by

Interpretation.

the simple annexation of such a basis.

70. However incapable of interpretation the nilfactorial and nilpotent expressions may appear, they are obviously an essential element of the calculus of linear algebras. The unwillingness to accept them has retarded the progress of discovery and the investigation of quantitative algebras. But the idempotent basis seems to be equally essential to actual interpretation. The purely nilpotent algebra may therefore be regarded as an ideal abstraction, which requires the introduction of an idempotent basis, to give it any position in the real universe.

In the subsequent investigations, therefore the purely nilpotent algebras must be regarded as the first steps towards the discovery of algebras of a higher degree and which rest upon an idempotent basis.

71. Sufficient preparation is now made for the investigation of special algebras.

The following notation will be adopted in these researches. Conformable with §9, the letters of the alphabet will be denoted by i, j, k, l, m , and n , to these letters will also be

Special investigation of linear
associative algebra.

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respectively assigned the numbers 1, 2, 3, 4, 5, and 6; and also their coefficients in an algebraic sum will be denoted by the letters a, b, c, d, e and f . Thus the product of any two letters will be expressed by an algebraic sum and below each coefficient will be written in order the numbers which are appropriate to the factors. Thus

$$j l = a_{24} i + b_{24} j + c_{24} k + d_{24} l + e_{24} m + f_{24} n,$$

while

$$l j = a_{42} i + b_{42} j + c_{42} k + d_{42} l + e_{42} m + f_{42} n$$

In the case of a square, only one number need be written below the coefficient, thus

$$k^2 = a_3 i + b_3 j + c_3 k + d_3 l + e_3 m + f_3 n$$

The investigation simply consists in the determination of the values of the coefficients, which correspond to every variety of linear algebra; and the resulting products can be arranged in a tabular form which may be called the multiplication-table of the algebra. Upon this table rests all the peculiarity of calculus. In each of the algebras, it admits of many transformations, and much corresponding speculation. The basis will be denoted by i .

72. The distinguishing of the successive cases by

Single algebra

the introduction of numbers, will explain itself, and is an indispensable protection from omission of important steps in the discussion.

Since ~~in~~ ^{Single Algebra} a single algebra, there is only one independent unit it requires no distinguishing letter. It is also obvious that there can be no single algebra which is not-associative and commutative. Single algebra has however two cases;

[1] when its unit is idempotent.

[2] when it is nilpotent

[1] The defining equation of this case is
 $i^2 = i.$

This algebra may be called (a_1) and its multiplication table is

$$\begin{array}{c|c} a_1 & i \\ \hline i & i \end{array}$$

[2] The defining equation of this case is
 $i^2 = 0.$

This algebra may be called (b_1) and its multiplication table is

$$\begin{array}{c|c} b_1 & i \\ \hline i & 0 \end{array}$$

Double algebra

47.

There are two cases of double algebra

[1] When it has an idempotent expression

[2] when it is nilpotent.

[1] The defining equation of this case is
 $i^2 = i.$

By §§ 41 and 50 there are two cases;

[1¹], When the other unit belongs to the first group

[1²], When it is of the second group.

The hypothesis that the other unit belongs to the third group is a virtual repetition of [1²].

[1¹]. The defining equations of this case are
 $ij = ji = j$

It follows from §§ 67 and 69 that there is a double algebra derived from (b_1) which may be called (a_2) , of which the multiplication table is

a_2	i	j
i	i	j
j	j	0

[1²] The defining equations of this case are by § 41
 $ij = j, ji = 0.$

whence by § 46

$$j = 0.$$

A double algebra is thus formed, which may be called (b_2) of which the multiplication table is

<u>Double algebra</u>		
c_2	i	j
i	i	j
j	0	0

[2]. The defining equation of this case is
 $i^n = 0$

in which n is the least power of i which vanishes
 there are two cases;

[2¹]. when $n = 3$.

[2²], when $n = 2$.

[2¹] The defining equation of this case is
 $i^3 = 0$.

and by § 60

$$i^2 = j.$$

This gives a double algebra which may be called
 (c_2) of which the multiplication table is

c_2	i	j
i	j	0
j	0	0

[2²] The defining equations of this case are
 $i^2 = j^2 = 0$

and it follows from §§ 64 and 65 that

$$ij = ji = 0$$

so that there is no pure algebra in this case.

Triple algebra.

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There are two cases

[1]. When there is an idempotent basis.

[2] When it is nilpotent

[1] The defining equation of this case is
 $i^2 = i.$

There are three cases by §§ 41, 50 and 51.

[1], When j and k are both in the first group.

[12], When j is in the first and k in the second group.

[13], When j is in the second and k in the third group.

The case, when j is in the first and k in the third group is a virtual repetition of [12].

[1]. The defining equations of this case are
 $ij = ji = j, \quad ik = ki = k.$

It follows from §§ 67 and 69 that the only algebra of this case may be derived from (c_2) and may be called (a_3) . Its multiplication table is

a_3	i	j	k
i	i	j	k
j	j	k	0
k	k	0	0

[12]. The defining equations of this case are

$$ji = ij = j, \quad ik = k, \quad ki = 0$$

whence by §§ 46 and 67

Triple algebra

$$j^2 = k^2 = kj = 0, \quad jk = c_{23} k,$$

$$j^2 k = 0 = c_{23} jk = c_{23}^2 k = c_{23} = jk,$$

and there is no pure algebra in this case.

[13] The defining equations of this case are

$$ij = j, \quad ki = k, \quad ji = ik = 0$$

whence by § 46

$$j^2 = k^2 = kj = 0, \quad jk = a_{23} i,$$

$$jkj = 0 = a_{23} j = a_{23} = jk,$$

and there is no pure algebra in this case.

[2]. The defining equation of this case is

$$i^n = 0,$$

in which n is the lowest power of i which vanishes

There are three cases:

[21]. When $n = 4$,

[22]. When $n = 3$,

[23]. When $n = 2$.

[21] The defining equation of this case is

$$i^4 = 0,$$

and by § 60

$$i^2 = j, \quad i^3 = k.$$

This gives a triple algebra, which may be called (b_3) of which the multiplication table is

Triple algebra

b_3	i	j	k
i	j	k	0
j	k	0	0
k	0	0	0

[2²]. The defining equation of this case is
 $i^3 = 0$

and by §§ 59 and 64, observing the exception

$$i^2 = j, \quad ik = 0$$

$$ki = b_3 j, \quad k^2 = b_3 j$$

There is no pure algebra when b_3 vanishes, and there are two cases;

[2²1]. When b_3 does not vanish,

[2³]. When b_3 vanishes.

[2²1] The defining equation of this case can, without loss of generality be reduced to
 $k^2 = j$.

This gives a triple algebra, which may be called (c_3) . of which the multiplication table is

c_3	i	j	k
i	j	0	0
j	0	0	0
k	aj	0	j

an interesting special example of this case occurs when $a = -2$ in which case

Triple algebra

$$i(k+i) = -j$$

$$(k+i)i = j$$

$$(k+i)^2 = 0$$

so that $k+i$ might be substituted for k , and in this form, the multiplication table of this algebra, which may be called (c_3) is

c_3	i	j	k
i	j	0	j
j	0	0	0
k	$-j$	0	0

[2]. The defining equation of this case is

$$k^2 = 0$$

and b_3 may be reduced to unity, without loss of generality, and there is a triple algebra, which may be called (d_3) , of which the multiplication table is

d_3	i	j	k
i	j	0	0
j	0	0	0
k	j	0	0

in this case

$$(i-k)k = 0$$

$$k(i-k) = i$$

$$(i-k)^2 = 0$$

so that $i-k$ may be substituted for i , and in

this form the multiplication table is

d_3	i	j	k
i	0	0	0
j	0	0	0
k	j	0	0

[23] The defining equations of this case are
 $i^2 = j^2 = k^2 = 0$

and by the principle of §§ 63 and 65, it may be assumed that

$$ij = -ji = ik = ki = 0,$$

$$jk = -kj = i$$

which give a triple algebra which may be called (e_3) of which the multiplication table is

e_3	i	j	k
i	0	0	0
j	0	0	i
k	0	$-i$	0

Quadruple algebra.

There are two cases

- [1]. When there is an idempotent basis.
- [2]. When it is nilpotent
- [1]. The defining equation of this case is

$$i^2 = i$$

There are six cases

- [1]. When j , k and l are all in the first group
- [12]. When j and k are in the first and l in the second group.
- [13]. When j is in the first and k and l in the second group.
- [14]. When j is in the first, k the second and l the third group.
- [15]. When j and k are in the second and l in the third group.
- [16]. When j is in the second, k the third, and l the fourth group.

The other cases are excluded by §§ 50 and 51, on the obvious fact that they are virtual repetitions of those which are given.

[1]. The defining equations of this case are

$$ij = ji = j, \quad ik = ki = k, \quad il = li = l.$$

and from §§ 60 and 69, the algebra $(b_3)(c_3)(d_3)$ and (e_3) give quadruple algebras which may be named respectively (a_4) , (b_4) , (c_4) and (d_4) of which the multiplication tables are

Quadruple algebras

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a_4	i	j	k	l
i	i	j	k	l
j	j	k	l	0
k	k	l	0	0
l	l	0	0	0

b_4	i	j	k	l
i	i	j	k	l
j	j	k	0	0
k	k	0	0	0
l	l	ak	0	k

c_4	i	j	k	l
i	i	j	k	l
j	j	k	0	0
k	k	0	0	0
l	l	k	0	0

d_4	i	j	k	l
i	i	j	k	l
j	j	0	0	0
k	k	0	0	j
l	l	0	j	0

Quadruple algebra

The special case (c'_3) gives a corresponding special case of (b_4) which may be called (b'_4) of which the multiplication table is

b'_4	i	j	k	l
i	i	j	k	l
j	j	k	0	k
k	k	0	0	0
l	l	$-k$	0	0

The second form of (d_3) gives a corresponding second form of (c_4) of which the multiplication table is

c_4	i	j	k	l
i	i	j	k	l
j	j	0	0	0
k	k	0	0	0
l	l	k	0	0

[12] The defining equations of this case are

$$ij = ji = j, \quad ik = ki = k, \quad il = l, \quad li = 0$$

and it follows from §§ 67 and 69, that (c_2) gives

$$j^2 = k, \quad jk = kj = k^2 = 0,$$

and from § 46

$$lj = lk = l^2 = 0, \quad jl = d_{24}l, \quad kl = d_{34}l,$$

whence

$$j^2l = kl = d_{24}jl = d_{24}^2l$$

$$jkl = d_{24}^2jl = d_{24}^3l = 0 = d_{24}jl = kl,$$

and there is no pure algebra in this case.

[13] The defining equations of this case are

$$ij = ji = j, \quad ik = k, \quad il = l, \quad ki = li = 0$$

which gives by ss 46 and 67

$$0 = j^2 = k^2 = kl = lk = l^2 = kj = lj$$

and it may be assumed that

$$jk = 0, \quad jl = k.$$

This gives a quadruple algebra which may be called (ϵ_4) of which the multiplication table is

ϵ_4	i	j	k	l
i	i	j	k	l
j	j	0	0	k
k	0	0	0	0
l	0	0	0	0

[14] The defining equations of this case are

$$ij = ji = j, \quad ik = k, \quad li = l, \quad ki = il = 0$$

which give by ss 46 and 67

$$0 = j^2 = jl = kj = k^2 = lk = l^2,$$

$$jk = c_{23}k, \quad lj = d_{42}l, \quad kl = a_{34}i + b_{34}j$$

$$0 = j^2k = c_{23}jk = c_{23}^2k = c_{23} = jk,$$

$$0 = lj^2 = d_{42}lj = d_{42}^2l = d_{42} = lj,$$

$$0 = jkl = a_{34}j = a_{34},$$

and b_{34} cannot be permitted to vanish, so that it does not lessen the generality to assume

$$kl = j$$

Quadruple algebra

This gives a quadruple algebra, which may be called $(\mathcal{F}_4)$ of which the multiplication table is

$\mathcal{F}_4$	i	j	k	l
i	i	j	k	0
j	j	0	0	0
k	0	0	0	j
l	l	0	0	0

[15] The defining equations of this case are
 $ij = j, ik = k, li = l, ji = ki = il = 0$
 which give by § 46

$$0 = j^2 = jk = kj = k^2 = lj = lk = l^2$$

$$jl = a_{24}i, \quad kl = a_{34}i$$

$$0 = jlj = a_{24}j = a_{24} = jl$$

$$0 = klk = a_{34}k = a_{34} = kl$$

and there is no pure algebra in this case.

[16] The defining equations of this case are
 $ij = j, ki = k, ji = ik = il = li = 0$
 which give by § 46

$$0 = j^2 = k^2 = kl = lj$$

$$jk = a_{23}i, \quad jl = b_{24}j, \quad kj = d_{32}l$$

$$lk = c_{43}k, \quad l^2 = d_4l,$$

$$j k j = a_{23}j = d_{32} b_{24}j, \quad a_{23} = b_{24} d_{32}$$

$$j l k = a_{23} b_{24} i = a_{23} c_{43} i, \quad j l^2 = b_{24}^2 j = b_{24} d_4 j$$

$$k j k = a_{23} k = c_{43} d_{32} k, \quad k j l = d_{32} d_4 l = b_{24} d_{32} l$$

Quadruple algebra.

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$$\ell k j = d_{32} d_4 \ell = c_{43} d_{32} \ell, \quad \ell^2 k = k_{43}^2 k = c_{43} d_4 k$$

$$0 = a_{23}(c_{43} - b_{24}) = b_{24}(b_{24} - d_{24}) = d_{32}(b_{24} - d_4) = d_{32}(c_{43} - d_4) = c_{43}(c_{43} - d_4)$$

There are two cases;

[161]. When d_{32} does not vanish,

[162]. When d_{32} vanishes.

[161]. The defining equation of this case can be reduced to

$$d_{32} = 1$$

which gives

$$a_{23} = b_{24} = c_{43} = d_4$$

There are two cases;

[161]. When d_4 does not vanish,

[1612]. When d_4 vanishes.

[1612]. The defining equation of this case can be reduced to

$$d_4 = 1$$

which gives

$$j k = i, \quad j \ell = j, \quad \ell k = k, \quad \ell^2 = \ell$$

and there is a quadruple algebra which may be called (g_4) of which the multiplication table is:

g_4	i	j	k	ℓ
i	i	j	0	0
j	0	0	i	j
k	k	ℓ	0	0
ℓ	0	0	k	ℓ

This is a form of quaternions.

Quadruple algebra

[1612]. The defining equation of this case is

$$d_4 = 0$$

which gives

$$jk = jl = lk = l^2 = 0$$

and there is a quadruple algebra which may be called (h_4) of which the multiplication table is

h_4	i	j	k	l
i	j	j	0	0
j	0	0	0	0
k	k	l	0	0
l	0	0	0	0

[162]. The defining equation of this case is

$$d_{32} = 0$$

which gives

$$d_{23} = 0$$

and there can be no pure algebra in this case

[2]. The defining equation of this case is

$$i^n = 0$$

There are four cases;

[21]. When $n = 5$

[22]. When $n = 4$

[23]. When $n = 3$

[24]. When $n = 2$

Quadruple algebra

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21. The defining equations of this case is

$$i^5 = 0$$

and by § 60

$$i^2 = j, \quad i^3 = k, \quad i^4 = l$$

This gives a quadruple algebra which may be called (i_4) of which the multiplication table is

i_4	i	j	k	l
i	j	k	l	0
j	k	l	0	0
k	l	0	0	0
l	0	0	0	0

[2²] The defining equation of this case is

$$i^4 = 0$$

and by § 59

$$i^2 = j, \quad i^3 = k$$

There are then by § 64 two quadruple algebras which may be called (j_4) and (k_4) of which the multiplication tables are

j_4	i	j	k	l
i	j	k	0	0
j	k	0	0	0
k	0	0	0	0
l	k	0	0	k

and

Quadruple algebra.

k_4	i	j	k	l
i	j	k	0	0
j	k	0	0	0
k	0	0	0	0
l	k	0	0	0

[23] The defining equation of this case is
 $i^3 = 0$

and by § 59

$$i^2 = j$$

and it may be assumed from the principle of § 63 that

$$ik = 0,$$

which gives

$$jk = 0.$$

There are two cases

[231]. when $il = k$

[232] when $il = 0$

[231]. The defining equation of this case is

$$il = k$$

which gives

$$jl = 0, \quad iki = ik^2 = ikl = 0$$

$$ki = b_{31}j + c_{31}k, \quad kj = c_{31}ki = b_{31}c_{31}j + c_{31}^2k$$

$$k^2 = b_3j + c_3k, \quad kji = c_{31}^2ki = 0 = c_{31} = kj$$

$$kl = b_{34}j + c_{34}k, \quad 0 = k^3 = c_3k^2 = c_3$$

$$ili = ki = b_{31}j, \quad li = b_{31}i + b_{41}j + c_{41}k$$

$$li^2 = lj = b_{31}j + b_{31}c_{41}j, \quad ilk = k^2 = b_3j$$

Quadruple algebra

$$lk = b_3 i + b_{32} j + c_{32} k$$

$$l^2 = a_4 i + b_4 j + c_4 k + d_4 l$$

$$0 = l^3 = a_4 k + c_4 kl + d_4 l^2 = a_4 li + b_4 lj + c_4 lk + d_4 l^2$$

$$0 = d_4, \quad il^2 = kl = a_4 j, \quad 0 = c_{34} = a_4 = kl$$

$$kil = k^2 = c_{31}, \quad kl = a_4 c_{31} j = 0$$

$$0 = kli = b_{31}, \quad ki = b_{31}^2 j = b_{31} = ki = lj$$

$$lil = lk = 0$$

There are two cases

[231³]. When c_{41} does not vanish,

[231²]. When c_{41} vanishes.

[231³] The defining equation of this case is
 $c_{41} \neq 0$

and if μ is determined by the equation

$$c_{41} \mu^2 + (c_4 + b_{41}) \mu = b_4,$$

we have

$$i(l + \mu i) = k + \mu j$$

$$(l + \mu i)^2 = (c_4 + \mu c_{41})(k + \mu j)$$

so that $l + \mu i$ and $k + \mu j$ may be substituted respectively for l and k , which is the same as to make $b_4 = 0$

and there are two cases;

[231³]. When c_4 does not vanish,

[231²]. When c_4 vanishes.

Quadruple algebra

[231³]. The defining equation of this case can be reduced to $l^2 = k$

This gives a quadruple algebra, which may be called (l_4) of which the multiplication table is

l_4	i	j	k	l
i	j	0	0	k
j	0	0	0	0
k	0	0	0	0
l	$aj+bk$	0	0	k

[231²2] The defining equation of this case is $l^2 = 0$

There are two cases;

[231²2]. When b_{41} does not vanish

[231²2]. When b_{41} vanishes.

[231²21]. The defining equation of this case is $b_{41} \neq 0$

There are two cases;

[231²21]. When $c_{41} + 1$ does not vanish

[231²21]. When $c_{41} + 1$ vanishes.

[231²21] The defining equation of this case is $c_{41} + 1 \neq 0$

so that

$$l \frac{b_{41} i + c_{41} l}{c_{41} + 1} = \frac{b_{41} j + b_{41} c_{41} k}{c_{41} + 1}$$

Quadruple algebra

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$$\frac{b_{41} + c_{41} l}{c_{41} + 1} \quad \frac{l = b_{41} k}{c_{41} + 1} \quad \frac{b_{41} + c_{41} l^2}{c_{41} + 1} \quad \frac{b_{41}^2 j + b_{41} c_{41} k}{c_{41} + 1}$$

so that the substitution of $\frac{b_{41} i + c_{41} l}{c_{41} + 1}$, $\frac{b_{41}^2 j + b_{41} c_{41} k}{c_{41} + 1}$ and $\frac{b_{41} k}{c_{41} + 1}$ respectively for i , j and k is the same as to assume

$$c_{41} = 0, \quad b_{41} = j$$

which reduces this case to $[2312]$

$[2312^2 212]$ The defining equation of this is easily reduced to $li = j - k$

This gives a quadruple algebra which may be called (m_4) of which the multiplication table is

m_4	i	j	k	l
i	j	0	0	k
j	0	0	0	0
k	0	0	0	0
l	$j - k$	0	0	0

The substitution of $i - l$ and $j - k$, respectively for i and j transforms this algebra into one of which the multiplication table is

m_4	i	j	k	l
i	0	0	0	k
j	0	0	0	0
k	0	0	0	0
l	j	0	0	0

$[2312^2 2^2]$ The defining equation of this case is

Quadruple algebra

$$li = c_{41} k$$

This gives a quadruple algebra, which may be called (n_4) of which the multiplication table is

n_4	i	j	k	l
i	j	0	0	k
j	0	0	0	0
k	0	0	0	0
l	ak	0	0	0

[2312]. The defining equation of this case is

$$li = b_{41} j$$

which gives

$$(l - b_{41} i) i = 0$$

so that the substitution of $l - b_{41} i$ for l passes this case virtually into [232].

[232]. The defining equation of this case is

$$il = 0$$

and it may be assumed that

$$ki = 0$$

$$0 = jl = kj = ik^2 = k^2 i = ikl = ili = ilk = lki = il^2$$

$$k^2 = b_3 j + c_3 k + d_3 l, \quad li = b_{41} j + c_{41} k + d_{41} l,$$

$$lj = d_{41} li, \quad 0 = lji = d_{41} lj = d_{41}^2 li = d_{41} = lj.$$

There are two cases;

[2321] When c_{41} does not vanish

[2322] When c_{41} vanishes.

Quadruple algebra

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[2321]. The defining equation of this case is easily reduced to $li = k$ which gives

$$\begin{aligned} 0 &= lik = k^2 = lil = kl \\ lk &= l^2i = a_4j + d_4k, \\ 0 &= l^3k = d_4l^2k = d_4^2lk = d_4, \quad lk = a_4j = l^2i \\ l^2 &= a_4i + b_4j + c_4k \\ 0 &= l^3 = a_4k + c_4a_4j = a_4 = lk \end{aligned}$$

There are two cases;

[2321³]. When c_4 does not vanish

[2321²]. When c_4 vanishes.

[2321²]. The defining equation of this case can be reduced to $c_4 = 1$

which gives a quadruple algebra which may be called (o_4) of which the multiplication table is

o_4	i	j	k	l
i	j	0	0	0
j	0	0	0	0
k	0	0	0	0
l	k	0	0	$k + a_4j$

[2321²] The defining equation of this case is $l^2 = b_4j$

There are two cases;

Quadruple algebra

$[232121]$. When b_4 does not vanish

$[23212^2]$. When b_4 vanishes

$[232121]$. The defining equation of this case can be reduced to $l^2 = f$

This gives a quadruple algebra which may be called (p_4) of which the multiplication table is

p_4	i	j	k	l
i	f	0	0	0
j	0	0	0	0
k	0	0	0	0
l	k	0	0	f

$[23212^2]$. The defining equation of this case is $l^2 = 0$

This gives a quadruple algebra which may be called (q_4) of which the multiplication table is

q_4	i	j	k	l
i	f	0	0	0
j	0	0	0	0
k	0	0	0	0
l	k	0	0	0

The defining equation of this case is $li = b_4 f$

and we have

$$k^2 = b_3 f + c_3 k + d_3 l$$

Quadruple algebra

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$$kl = b_{34}j + c_{34}k + d_{34}l$$

$$lk = b_{43}j + c_{43}k + d_{43}l$$

$$l^2 = b_4j + c_4k + d_4l$$

so that there can be no pure algebra in this case if b_4 vanishes, and it may be assumed without loss of generality that $li = j$

There are two cases

[232²1] When d_3 does not vanish

[232³]. When d_3 vanishes.

[232²1] The defining equation of this case can be reduced to $k^2 = l$ which gives

$$0 = k^3 = kl = lk = k^2l = l^2$$

and there is no pure algebra in this case

[232³]. The defining equation of this case is $d_3 = 0$

which gives

$$0 = k^3 = c_3k^2 = c_3, \quad k^2 = b_3j$$

$$0 = k^2l = c_{34}k^2 + d_{34}kl = d_{34} = b_3c_{34}$$

$$0 = lk^2 = c_{43}k^2 + d_{43}kl = d_{43} = b_3c_{43}$$

There are two cases

[232³1] When b_3 does not vanish

[232⁴] When b_3 vanishes.

Quadruple algebra

[232³1]. The defining equation of this case can be reduced to

$$k^2 = j$$

which gives

$$0 = c_{34} = c_{43}, \quad kl = b_{34}j, \quad lk = b_{43}j$$

$$k(l - b_{34}j) = 0$$

so that $l - b_{34}j$ can be substituted for l without loss of generality, which is the same to assume

$$kl = 0$$

which gives

$$0 = l^3 = d_4 l^2 = d_4 = c_4 lk = c_4 b_{43} = l^2 k = c_4$$

and there is no pure algebra in this case

[232⁴] The defining equation of this case is

$$k^2 = 0$$

which gives

$$0 = lj = l^2 i = d_4 j = d_4$$

$$0 = kl^2 = c_{34}kl = c_{34}, \quad kl = b_{34}j$$

$$0 = l^2 k = c_{43}lk = c_{43}, \quad lk = b_{43}j$$

and there can be no pure algebra if c_4 vanishes, so that it may be assumed, without loss of generality, that

$$l^2 = k$$

which gives

$$0 = l^3 = lk = kl$$

This gives a quadruple algebra, which may be called (r_4) of which the multiplication table is

Quadruple algebra

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r_4	i	j	k	l
i	j	0	0	0
j	0	0	0	0
k	0	0	0	0
l	j	0	0	k

[24] The defining equations of this case are

$$i^2 = j^2 = k^2 = l^2 = 0$$

and it may be assumed from §§ 63 and 65 that

$$ij = k = -ji, \quad il = li = 0$$

which give

$$0 = ik = ki = jk = kj = kl = lk$$

$$0 = ij l = b_{24} k = b_{24} = j^2 l = -a_{24} k + d_{24} j l = d_{24} = a_{24}$$

$$j l = -l j = c_{24} k$$

so that there is no pure algebra in this case.

There are two cases:

[1]. When there is an idempotent basis,

[2]. When the algebra is nilpotent,

[1]. The defining equation of this case is

$$i^2 = i$$

Quintuple Algebra

There are eleven cases;

- [1¹]. When j, k, l and m are all in the first group.
- [12]. When j, k and l are in the first and m in the second group.
- [13]. When j and k are in the first and l and m in the second group.
- [14]. When j and k are in the first, l in the second, and m in the third group.
- [15]. When j is in the first and k, l and m in the second group.
- [16]. When j is in the first, k and l in the second, and m in the third group.
- [17]. When j is in the first, k in the second, l in the third and m in the fourth group.
- [18]. When j, k and l are in the second and m in the third group.
- [19]. When j and k are in the second and l and m in the third group.
- [10¹]. When j and k are in the second, l in the third and m in the fourth group.
- [11¹]. When j is in the second, k in the third, & l and m in the fourth group. [1²]. The defining equations of this case are $ij = ji = j$, $ik = ki = k$, $il = li = l$, $im = mi = m$.
The algebras deduced by §69 from (i_4) to (r_4) may be

Quintuple algebra

named (a_s) to (j_s) , and their multiplication tables are respectively

a_s	i	j	k	l	m
i	i	j	k	l	m
j	j	k	l	m	o
k	k	l	m	o	o
l	l	m	o	o	o
m	m	o	o	o	o

b_s	i	j	k	l	m
i	i	j	k	l	m
j	j	k	l	o	o
k	k	l	o	o	o
l	l	o	o	o	o
m	m	l	o	o	l

c_s	i	j	k	l	m
i	i	j	k	l	m
j	j	k	l	o	o
k	k	l	o	o	o
l	l	o	o	o	o
m	m	l	o	o	o

d_s	i	j	k	l	m
i	i	j	k	l	m
j	j	k	o	o	l
k	k	o	o	o	o
l	l	o	o	o	o
m	m	$ak+bl$	o	o	l

Quintuple algebra

ϵ_5	i	j	k	l	m
i	i	j	k	l	m
j	j	k	0	0	l
k	k	0	0	0	0
l	l	0	0	0	0
m	m	$k-l$	0	0	0

or in the second form

ϵ_5	i	j	k	l	m
i	i	j	k	l	m
j	j	0	0	0	l
k	k	0	0	0	0
l	l	0	0	0	0
m	m	k	0	0	0

f_5	i	j	k	l	m
i	i	j	k	l	m
j	j	k	0	0	l
k	k	0	0	0	0
l	l	0	0	0	0
m	m	al	0	0	0

g_5	i	j	k	l	m
i	i	j	k	l	m
j	j	k	0	0	0
k	k	0	0	0	0
l	l	0	0	0	0
m	n	l	0	0	$l+ak$

Quintuple algebra

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h_5	i	j	k	l	m
i	i	j	k	l	m
j	j	k	0	0	0
k	k	0	0	0	0
l	l	0	0	0	0
m	m	l	0	0	k

i_5	i	j	k	l	m
i	i	j	k	l	m
j	j	k	0	0	0
k	k	0	0	0	0
l	l	0	0	0	0
m	m	l	0	0	0

j_5	i	j	k	l	m
i	i	j	k	l	m
j	j	k	0	0	0
k	k	0	0	0	0
l	l	0	0	0	0
m	m	k	0	0	l

[12]. The defining equations of this case are

$ij = ji = j$, $ik = ki = k$, $il = li = l$, $im = m$, $mi = 0$
which gives by § 46

$$0 = mj = mk = ml = m^2$$

and if A is any expression belonging to the first group but not involving i , we have the form

Quintuple algebra

$$A m = a m$$

and by § 67, A is nilpotent, so that there is some power n which gives $0 = A^n = A^n m = a A^{n-1} m = a^n m = a = A m$

$$0 = j m = k m = l m$$

and there is no pure algebra in this case.

[13]. The defining equations of this case are

$$ij = ji = j, \quad ik = ki = k, \quad il = l, \quad im = m, \\ li = mi = 0 \quad \text{which gives by § 46}$$

$$0 = lj = lk = l^2 = lm = mj = mk = ml = m^2$$

and it may be assumed from (a_3) by § 69 that

$$j^2 = k, \quad j^3 = 0.$$

It may also be assumed that

$$jl = m, \quad kl = jm = 0$$

This gives a quintuple algebra which may be called (k_s) of which the multiplication table is

k_s	i	j	k	l	m
i	i	j	k	l	m
j	j	k	0	m	0
k	k	0	0	0	0
l	0	0	0	0	0
m	0	0	0	0	0

[14]. The defining equations of this case are

$$ij = ji = j, \quad ik = ki = k, \quad il = l, \quad mi = m, \quad li = im = 0 \\ \text{which gives by § 46}$$

$$0 = jm = km = lj = lk = l^2 = ml = m^2$$

It may be assumed from § 69 and (a_3) that

$$j^2 = k, \quad j^3 = 0$$

which gives

$$0 = jl = kl = mj = mk = jlm = a_{45}j + b_{45}k = a_{45} = b_{45}$$

$$lm = c_{45}k$$

and there is no pure algebra in this case

[15]. The defining equations of this case are

$$ij = ji = j, \quad ik = k, \quad il = l, \quad im = m, \quad ki = li = mi = 0,$$

which give by §§ 46 and 67

$$0 = j^2 = kj = k^2 = kl = km = lj = lk = l^2 = lm = mj = mk = ml = m^2$$

It may be assumed that

$$jk = l, \quad jm = 0$$

which gives

$$jl = 0$$

and there is no pure algebra in this case

[16]. The defining equations of this case are

$$ij = ji = j, \quad ik = k, \quad il = l, \quad mi = m, \quad ki = li = im = 0$$

which give by §§ 46 and 67

$$0 = j^2 = jm = kj = k^2 = kl = lj = lk = l^2 = mj = mk = ml = m^2$$

$$km = a_{35}i + b_{35}j, \quad lm = a_{45}i + b_{45}j$$

and it may be assumed that

$$jk = d_{23}l, \quad jl = 0$$

and d_{23} cannot vanish in the case of a pure algebra,

Quintuple algebra

so that it is no loss of generality to assume
 $j k = l$

which gives

$$j k m = l m = a_{35} j$$

There are two cases

[161]. When a_{35} does not vanish

[162]. When a_{35} vanishes

[161]. The defining equation of this case can be reduced to
 $a_{35} = 1$

which gives

$$l m = j, \quad k m = i + b_{35} j$$

and $i + b_{35} j$ can be substituted for i , and this gives a quintuple algebra which may be called (L_s) of which the multiplication table is

L_s	i	j	k	l	m
i	i	j	k	l	0
j	i	0	l	0	0
k	0	0	0	0	i
l	0	0	0	0	j
m	m	0	0	0	0

[162]. The defining equation of this case is

$$a_{35} = 0$$

which gives $k m = b_{35} j$, $l m = 0$

and b_{35} cannot vanish in the case of a pure algebra,

so that it is no loss of generality to assume

$$km = j$$

This gives a quintuple algebra which may be called (m_5) of which the multiplication table is

m_5	i	j	k	l	m
i	i	j	k	l	0
j	j	0	l	0	0
k	0	0	0	0	j
l	0	0	0	0	0
m	m	0	0	0	0

[17]. The defining equations of this case are

$ij = ji = j$, $ik = k$, $li = l$, $ki = il = im = mi = 0$
which gives by §§ 46 and 67

$$0 = j^2 = jk = jl = jm = kj = k^2 = lj = l^2 = lm = mj = mk$$

$$kl = a_{34}i + b_{34}j, \quad km = c_{35}k, \quad lk = c_{43}m$$

$$ml = d_{54}l, \quad m^2 = e_5m$$

$$0 = jkl = a_{34}j = a_{34}$$

$$lkl = b_{34}lj = 0 = c_{43}ml = c_{43}d_{54}$$

$$klk = b_{34}jk = 0 = c_{43}km = c_{43}c_{35}$$

$$lkm = c_{35}lk = c_{43}m^2 = c_{35}c_{43}m = 0 = c_{43}e_5$$

$$kml = d_{54}kl = c_{35}kl, \quad (d_{54} - c_{35})b_{34} = 0$$

$$km^2 = e_5km = c_{35}kn, \quad (e_5 - c_{35})c_{35} = 0$$

$$m^2l = e_5ml = d_{54}ml, \quad (e_5 - d_{54})d_{54} = 0$$

There are two cases;

Quintuple algebra

[171] When $c_5 = 1$ [172] When $c_5 = 0$

[171]. The defining equation of this case is

$$m^2 = m$$

which gives

$$0 = c_{43} = lk$$

There can be no pure algebra, if either of the quantities b_{34} , c_{35} or d_{54} vanish, and there is no loss of generality in assuming

$$kl = j, \quad km = k, \quad ml = l$$

This gives a quintuple algebra which may be called (n_5) of which the multiplication table is

n_5	i	j	k	l	m
i	i	j	k	0	0
j	j	0	0	0	0
k	0	0	0	j	k
l	l	0	0	0	0
m	0	0	0	l	m

[172]. The defining equation of this case is

$$m^2 = 0$$

which gives

$$0 = c_{35} = d_{54} = km = ml$$

and there can be no pure algebra if either b_{34} or c_{43} vanishes, and it may be assumed that

$$kl = j, \quad lk = m$$

Quintuple algebra

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This gives a quintuple algebra which may be called o_5 of which the multiplication table is

o_5	i	j	k	l	m
i	i	j	k	0	0
j	j	0	0	0	0
k	0	0	0	j	0
l	l	0	m	0	0
m	0	0	0	0	0

[18]. The defining equations of this case are
 $ij = j$, $ik = k$, $il = l$, $mi = m$, $ji = ki = li = im = 0$
 which gives by § 46,

$0 = j^2 = jk = jl = kj = k^2 = kl = lj = lk = l^2 = mj = mk = ml = m^2$,
 and if A is any expression of the second group,
 there is the form $Am = ai$
 which gives

$$0 = Amj = aj = a = Am = jm = km = lm$$

and there is no pure algebra in this case.

[19]. The defining equations of this case are
 $ij = j$, $ik = k$, $li = l$, $mi = m$, $il = im = ji = ki = 0$
 which give by § 46

$0 = j^2 = jk = kj = k^2 = lj = lk = l^2 = lm = mj = mk = ml = m^2$
 and if A is an expression of the second group and
 B one of the third, there is the form $AB = ai$
 which gives

Quintuple algebra

$$0 = ABj = aj = a = AB = jl = jm = kl = lm$$

and there is no pure algebra in this case.

[10']. The defining equations of this case are

$$ij = j, ik = k, li = l, ji = ki = il = im = mi = 0$$

which gives by § 46

$$0 = j^2 = jk = kj = k^2 = l^2 = lm = mj = mk$$

and it is obvious that we may assume

$$jl = 0$$

and we have

$$jm = b_{25}j + c_{25}k,$$

$$kl = a_{34}i, \quad km = b_{35}j + c_{35}k.$$

$$lj = e_{42}m, \quad lk = e_{43}m$$

$$ml = d_{54}l, \quad m^2 = d_5m.$$

$$0 = d_{54}jl = jml = c_{25}kl = a_{34}c_{25}i = a_{34}c_{25}$$

There are two cases;

[10'1]. When a_{34} does not vanish

[10'2]. When a_{34} vanishes

[10'1]. The defining equation of this case can be reduced to $kl = i$

which gives $c_{25} = 0, \quad jm = b_{25}j$

There are two cases;

[10'1²]. When $d_5 = 1$

[10'12] When d_5 vanishes.

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[10'1']. The defining equation of this case is

$$m^2 = m$$

and it would be virtually proceeding to a subsequent case unless we assume

$$jm = j, \quad ml = l, \quad km = k$$

which gives

$$0 = jlj = \epsilon_{42} jm = \epsilon_{42} = lj$$

$$0 = jlk = \epsilon_{43} jm = \epsilon_{43} = lk$$

and virtually advances this case to [10'2]

[10'12] The defining equation of this case is

$$m^2 = 0$$

which gives

$$0 = jm^2 = b_{25} jm = b_{25} = jm$$

$$0 = m^2l = d_{52} ml = a_{54} = ml$$

$$0 = km^2 = c_{35} km = c_{35}, \quad km = b_{35}j$$

$$lkl = li = l = \epsilon_{43} ml = 0$$

which is impossible and this case disappears

[10'2] The defining equation of this case is

$$kl = 0$$

There are two cases;

[10'21] When $d_5 = 1$

[10'2']. When d_5 vanishes

[10'21]. The defining equation of this case is

$$m^2 = m$$

and if we would not virtually proceed to a

Quintuple algebra
subsequent case, we must assume

$j m = j, k m = k, m l = l$
and it is no loss of generality to assume
 $l j = 0$

so that there is no pure algebra in this case

[10'2'] The defining equation of this case is
 $m^2 = 0$

which gives $0 = m^2 l = d_{54} m l = d_{54} = m l$
and we may assume

$$c_{25} = 0$$

which gives

$$0 = j m^2 = b_{25} j m = b_{25} = j m$$

$$0 = k m^2 = c_{35} k m = c_{35}, \quad k m = b_{35} j$$

$$0 = e_{43} m^2 = l k m = b_{35} e_{42} m = b_{35} e_{42}$$

and we have without loss of generality

$$l j = 0, \quad k m = j, \quad l k = m.$$

This gives a quintuple algebra which may be called
(p_5) of which the multiplication Table is

p_5	i	j	k	l	m
i	i	j	k	0	0
j	0	0	0	0	0
k	0	0	0	0	j
l	l	0	m	0	0
m	0	0	0	0	0

[11']. The defining equations of this case are
 $ij = j, ki = k, ji = ik = il = im = li = mi$
 which give by § 46

$$\begin{aligned} 0 &= j^2 = k^2 = kl = km = lj = mj \\ jk &= a_{23}i, jl = b_{24}j, jm = b_{25}j \\ kj &= d_{32}l + c_{32}m \\ lk &= c_{43}k, mk = c_{53}k \end{aligned}$$

There are two cases;

[11'1]. When l is the idempotent base of the fourth group

[11'2]. When the fourth group is nilpotent,

[11'1]. The defining equation of this case is
 $l^2 = l$

There are two cases;

[11'1²]. When m is in the second subsiding group of the fourth group.

[11'12]. When m is in the fourth subsiding group of the fourth group.

[11'1²]. The defining equations of this case are
 $lm = m, ml = 0$

which give

$$\begin{aligned} 0 &= m^2 = jm^2 = b_{25}jm = b_{25} = jm \\ 0 &= m^2k = c_{53}mk = c_{53} = mk \end{aligned}$$

and a_{23} cannot vanish in a pure algebra so that we may assume $jk = i$

which gives $kjk = k = d_{32}c_{43}k, jkj = j = d_{32}b_{24}j$

Quintuple algebra

$$1 = d_{32} \quad c_{43} = d_{32} \quad b_{24}$$

$$jl = jl^2 = b_{24} jl, \quad b_{24}^2 = b_{24} = 1$$

$$lk = l^2 k = c_{43} lk, \quad c_{43}^2 = c_{43} = 1 = d_{32}$$

$$jl = j, \quad lk = k$$

$$kjl = l = kj$$

and there is no pure algebra in this case

[11'12]. The defining equations of this case are

$$lm = ml = 0$$

which give

$$0 = jlm = b_{24} jm = b_{24} b_{25} j = b_{24} b_{25}$$

$$0 = lmk = c_{53} lk = c_{43} c_{53} k = c_{43} c_{53}$$

$$kjl = d_{32} l = b_{24} kj = b_{24} d_{32} l + b_{24} e_{32} m$$

$$lkj = d_{32} l = c_{43} kj = c_{43} d_{32} l + c_{43} e_{32} m$$

$$kjm = e_{32} m^2 = b_{25} kj = b_{25} d_{32} l + b_{25} e_{32} m$$

$$mkj = e_{32} m^2 = c_{53} kj = c_{53} d_{32} l + c_{53} e_{32} m$$

$$d_{32} = b_{24} d_{32} = c_{43} d_{32}, \quad 0 = b_{25} d_{32} = e_{53} d_{32} = b_{24} e_{32} = c_{43} e_{32}$$

There are two cases;

[11'12i]. When m is idempotent

[11'12²]. When m is nilpotent

[11'12i]. The defining equation of this case is

$$m^2 = m$$

which gives

$$e_{32} = c_{53} e_{32} = b_{25} e_{32}$$

and it may be assumed that $b_{24} = 0$

Quintuple algebra

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But if the algebra is then regarded as having l for its idempotent basis, it is evident from § 50, that the bonds required for a pure algebra are wanting so that there is no pure algebra in this case.

[11'12²]. The defining equation of this case is

$$m^2 = 0$$

which gives

$$0 = j m^2 = b_{25} j m = b_{25}^2 j = b_{25} = j m$$

$$0 = m^2 k = e_{53} m k = c_{53}^2 k = e_{53} = m k$$

$$1 = b_{24} = c_{43}, j l = j, l k = k, 0 = e_{32}$$

and there is no pure algebra in this case.

[11'2]. The defining equation of this case is

$$l^n = 0$$

in which n is 2 or 3. We must then have

$$0 = l m = m l = m^2$$

which give

$$0 = j l^3 = b_{24} j l^2 = b_{24}^2 j l = b_{24} = j l = j m = l k = m k$$

$$0 = k j k = a_{23} k = a_{23} = j k$$

and there is no pure algebra in this case.

[2]. The defining equation of this case is

$$i^n = 0$$

There are five cases;

Quintuple algebra

[21]. When $n = 6$ [2²]. When $n = 5$ [23]. When $n = 4$ [24]. When $n = 3$ [25]. When $n = 2$ [21]. The defining equation of this case is
 $i^6 = 0$

and by § 60

$$i^2 = j, i^3 = k, i^4 = l, i^5 = m$$

This gives a quintuple algebra which may be called (q_5) of which the multiplication table is

q_5	i	j	k	l	m
i	j	k	l	m	0
j	k	l	m	0	0
k	l	m	0	0	0
l	m	0	0	0	0
m	0	0	0	0	0

[2²]. The defining equation of this case is
 $i^5 = 0$

and by § 59

$$i^2 = j, i^3 = k, i^4 = l$$

There are then by § 64 two quintuple algebras which may be called (r_5) and (s_5) of which the multiplication tables are

Quintuple algebra

r_s	i	j	k	l	m
i	i	k	l	0	0
j	k	l	0	0	0
k	l	0	0	0	0
l	0	0	0	0	0
m	l	0	0	0	l

s_s	i	j	k	l	m
i	j	k	l	0	0
j	k	l	0	0	0
k	l	0	0	0	0
l	0	0	0	0	0
m	l	0	0	0	0

[23]. The defining equation of this case is

$$i^4 = 0$$

and by § 59

$$i^2 = j, \quad i^3 = k$$

and it may be assumed from the principle of § 63 that

$$il = 0$$

which gives

$$0 = jl = kl = ili = il^2 = ilm$$

$$li = c_{41}k + d_{41}l + e_{41}m, \quad l^2 = c_4k + d_4l + e_4m, \quad lm = c_{45}k + d_{45}l + e_{45}m$$

There are two cases;

Quintuple algebra

[231]. When $im = l$ [232]. When $im = 0$ [231]. The defining equation of this case is
 $im = l$

$$0 = jm = km = jmi = jml = jm^2 = \varepsilon_{41} = \varepsilon_4 = \varepsilon_{45}$$

$$l^2 = d_{41} li, 0 = li^4 = d_{41} li^3 = d_{41}^2 li^2 = d_{41}^3 li = d_{41}^4 lj = lk$$

$$l^3 = d_4 l^2, 0 = l^4 = d_4 l^3 = d_4, li = c_{41} k, l^2 = c_4 k$$

$$lm = c_{45} k + d_{45} l,$$

$$imi = li = c_{41} k, mi = c_{41} j + c_{51} k + d_{51} l,$$

$$mj = c_{41} (1 + d_{51}) k, mk = 0$$

$$iml = l^2 = c_4 k, ml = c_4 j + c_{54} k + d_{54} l$$

$$im^2 = lm = c_{45} k + d_{45} l, m^2 = c_{45} j + c_5 k + d_5 l + d_{45} m$$

$$0 = m^4 = d_{45}, lim = l^2 = c_{41} km = 0 = mli = d_{54} c_{41} k = d_{54} c_{41}$$

$$0 = mlm = c_{54} lm = d_{54} c_{45}, 0 = m^2 l = d_{54} ml = d_{54}$$

There are two cases;

[231²]. When c_{41} does not vanish.[231²2]. When c_{41} vanishes.[231²]. The defining equation of this case is
reducible to $li = k$

There are two cases;

[231³]. When c_{45} does not vanish,[231³2]. When c_{45} vanishes[231³]. The defining equation of this case can be
reduced to $lm = k$

Quintuple algebra

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which gives

$$m^2 i = k + d_{s'} k + d_{s'}^2 k = k + d_s k, \quad d_s = d_{s'} + d_{s'}^2$$

$$m^3 = k + d_{s'} k + d_s d_{s'} k = d_s k, \quad d_{s'}^3 = -1$$

and if τ is one of the imaginary cube roots of -1 , there are two cases;

$$[231^4]. \text{ When } d_{s'} = \tau,$$

$$[231^3 2]. \text{ When } d_{s'} = -1$$

$$[231^4]. \text{ The defining equation of this case is } d_{s'} = \tau$$

which gives

$$i(m - c_{s'} l) = l \quad l(m - c_{s'} l) = k$$

$$(m - c_{s'} l) i = j + \tau l, \quad (m - c_{s'} l) j = (1 + \tau) k$$

$$(m - c_{s'} l) k = 0, \quad (m - c_{s'} l) l = \tau k$$

$$(m - c_{s'} l)^2 = j + [c_s - c_{s'}(1 + \tau)] k + (i\tau - 1) l$$

so that the substitution of $m - c_{s'} l$ for m is the same as to make

$$c_{s'} = 0$$

There are two cases;

$$[231^5]. \text{ When } c_s \text{ does not vanish}$$

$$[231^4 2]. \text{ When } c_s \text{ vanishes.}$$

$[231^5].$ The defining equation of this case can be reduced to $c_s = 1$

There is then a quintuple algebra which may be called (t_s) of which the multiplication table is

Quintuple algebra

t_s	i	j	k	l	m
i	j	k	0	0	l
j	k	0	0	0	0
k	0	0	0	0	0
l	k	0	0	0	k
m	$j+rl$	$(1+r)k$	0	rk	$j+k+(ir-1)l$

[231⁴2]. The defining equation of this case is

$$c_s = 0$$

There is, then a quintuple algebra which may be called (u_s) of which the multiplication table is

u_s	i	j	k	l	m
i	j	k	0	0	l
j	k	0	0	0	0
k	0	0	0	0	0
l	k	0	0	0	k
m	$j+rl$	$(1+r)l$	0	rk	$j+(ir-1)l$

[231³2]. The defining equation of this case is

$$d_{s1} = -1$$

which gives

$$d_s = 0, \quad i(m - c_{s1}k) = l, \quad l(m - c_{s1}k) = k$$

$$(m - c_{s1}k)i = j - l, \quad (m - c_{s1}k)l = -k, \quad (m - c_{s1}k) = j + c_s k$$

so that the substitution of $m - c_{s1}k$ for m is the same as to make

$$c_{s1} = 0$$

Quintuple algebra

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There are two cases;

$[231^321]$. When c_5 does not vanish

$[231^32^2]$. When c_5 vanishes.

$[231^321]$. The defining equation of this case can be reduced to $c_5 = 1$

There is a quintuple algebra which may be called (v_5) of which the multiplication table is

v_5	i	j	k	l	m
i	j	k	0	0	l
j	k	0	0	0	0
k	0	0	0	0	0
l	k	0	0	0	k
m	$j-l$	0	0	$-k$	$j+k$

$[231^32^2]$. The defining equation of this case is $c_5 = 0$

This gives a quintuple algebra which may be called (w_5) of which the multiplication table is

w_5	i	j	k	l	m
i	j	k	0	0	l
j	k	0	0	0	0
k	0	0	0	0	0
l	k	0	0	0	k
m	$j-l$	0	0	$-k$	j

$[231^2 2]$ The defining equation of this case is
 $l m = 0$

which gives

$$m l = 0, \quad m^2 = c_s k + d_s l$$

$$d_s = 1 + d_{s_1}$$

and c_{s_1} may be made to vanish without loss of generality.

There are three cases;

$[231^2 21]$. When d_{s_1} and $d_{s_1} + 1$ do not vanish,

$[231^2 2^2]$. When $d_{s_1} + 1$ vanishes.

$[231^2 2\bar{3}]$. When d_{s_1} vanishes.

$[231^2 2\bar{1}]$. The defining equation of this case is
 $d_{s_1} \neq 0$.

There are two cases;

$[231^2 21^2]$. When c_s does not vanish,

$[231^2 21\bar{2}]$. When c_s vanishes,

$[231^2 2\bar{1}^2]$. The defining equation of this case can always be reduced to

$$c_s = 1$$

This gives a quintuple algebra which may be called (x_s) of which the multiplication table is

Quintuple algebra

x_s	i	j	k	l	m
i	j	k	0	0	l
j	k	0	0	0	0
k	0	0	0	0	0
l	k	0	0	0	0
m	$j+al$	$(1+a)k$	0	0	$k+(1+a)l$

$[231^2 212]$. The defining equation of this case is

$$c_s = 0$$

This gives a quintuple algebra which may be called (y_s) of which the multiplication table is

y_s	i	j	k	l	m
i	j	k	0	0	0
j	k	0	0	0	0
k	0	0	0	0	0
l	k	0	0	0	0
m	$j+al$	$(1+a)k$	0	0	$(1+a)l$

$[231^2 2^2]$. The defining equation of this case is

$$d_{s1} = -1$$

which gives

$$mi = j - l, \quad mj = 0, \quad m^2 = c_s k$$

There are two cases;

$[231^2 2^2 1]$ When c_s does not vanish

$[231^2 2^3]$ When c_s vanishes

Quintuple algebra

$[231^2 2^2 1]$. The defining equation of this case can be reduced to

$$m^2 = k$$

This gives a quintuple algebra which may be called (z_5) of which the multiplication table is

z_5	i	j	k	l	m
i	j	k	0	0	l
j	k	0	0	0	0
k	0	0	0	0	0
l	k	0	0	0	0
m	$j-l$	0	0	0	k

$[231^2 2^3]$. The defining equation of this case is

$$m^2 = 0$$

This gives a quintuple algebra which may be called (aa_5) of which the multiplication table is

aa_5	i	j	k	l	m
i	j	k	0	0	l
j	k	0	0	0	0
k	0	0	0	0	0
l	k	0	0	0	0
m	$j-l$	0	0	0	0

$[231^2 2^3]$. The defining equation of this case is

$$mi = j$$

which gives

$$0 = (l-j)i = (m-i)i$$

so that by the substitution of $l-j$ for l and $m-i$ for m this case would virtually be reduced to $[232]$.

$[2312]$. The defining equation of this case is

$$li = 0$$

which gives

$$mj = 0, \quad m i m = m e = d_{s1} l m,$$

$$c_4 = 0, \quad c_{s4} = d_{s1} c_{45}$$

$$m^2 i = d_{s1} m l = c_{45} k, \quad c_{45} = d_{s1} c_{s4}$$

$$m^3 = d_s l m = d_s m l, \quad d_s (c_{s4} - c_{45}) = 0$$

There are two cases:

$[23121]$. When d_s does not vanish,

$[2312^2]$. When d_s vanishes,

$[23121]$. The defining equation of this case can be reduced to

$$d_s = 1$$

which gives

$$c_{45} = c_{s4}$$

and it may be assumed without loss of generality that

$$c_s = 0$$

There are two cases;

$[23121^2]$. When c_{45} does not vanish,

$[231212]$. When c_{45} vanishes.

Quintuple algebra

$[23121^2]$. The defining equation of this case can be reduced to

$$lm = ml = k$$

which gives

$$d_{s1} = 1$$

There are two cases;

$[23121^3]$. When c_{s1} does not vanish,

$[23121^2]$. When c_{s1} vanishes.

$[23121^3]$. The defining equation of this case can be reduced to

$$c_{s1} = 1$$

This gives a quintuple algebra which may be called (ab_s) of which the multiplication table is

ab_s	i	j	k	l	m
i	j	k	0	0	l
j	k	0	0	0	0
k	0	0	0	0	0
l	0	0	0	0	k
m	$k+l$	0	0	k	$j+l$

$[23121^2]$. The defining equation of this case is

$$c_{s1} = 0$$

This gives a quintuple algebra which may be called (ac_s) of which the multiplication table is

Quintuple algebra

ac_s	i	j	k	l	m
i	j	k	0	0	l
j	k	0	0	0	0
k	0	0	0	0	0
l	0	0	0	0	k
m	l	0	0	k	$j+l$

$[231212]$. The defining equation of this case is
 $ml = lm = 0$

There are two cases;

$[2312121]$. When c_{s1} does not vanish,

$[231212^2]$. When c_{s1} vanishes.

$[2312121]$. The defining equation of this case can be reduced to $c_{s1} = 1$

This gives a quintuple algebra which may be called (ad_s) of which the multiplication table is

ad_s	i	j	k	l	m
i	j	k	0	0	l
j	k	0	0	0	0
k	0	0	0	0	0
l	0	0	0	0	0
m	$k+al$	0	0	0	l

$[231212^2]$. The defining equation of this case is
 $c_{s1} = 0$

Quintuple algebra

This gives a quintuple algebra which may be called $(a\epsilon_5)$ of which the multiplication table is

$a\epsilon_5$	i	j	k	l	m
i	j	k	0	0	l
j	k	0	0	0	0
k	0	0	0	0	0
l	0	0	0	0	0
m	al	0	0	0	l

$[2312^2]$. The defining equation of this case is $d_5 = 0$

There are two cases

$[2312^21]$. When c_{45} does not vanish

$[2312^2]$. When c_{45} vanishes.

$[2312^21]$. The defining equation of this case can be reduced to $lm = k$ which gives

$$c_{45} = d_{s1}, \quad d_{s1}^2 = 1$$

There are two cases;

$[2312^21^2]$. When $d_{s1} = 1$

$[2312^212]$. When $d_{s1} = -1$

$[2312^21^2]$. The defining equation of this case is $d_{s1} = 1$

which gives

$$c_{s4} = 1, \quad ml = k$$

There are two cases;

$[2312^21^3]$. When c_{s1} does not vanish,

$[2312^21^22]$. When c_{s1} vanishes.

$[2312^21^3]$. The defining equation of this case can be reduced to $c_{s1} = 1$

This gives a quintuple algebra which may be called (af_s) of which the multiplication table is

af_s	i	j	k	l	m
i	j	k	o	o	l
j	k	o	o	o	o
k	o	o	o	o	o
l	o	o	o	o	k
m	$k+l$	o	o	k	$j+ak$

$[2312^21^22]$. The defining equation of this case is $c_{s1} = 0$

There are two cases;

$[2312^21^221]$. When c_s does not vanish,

$[2312^21^22^2]$. When c_s vanishes.

$[2312^21^221]$. The defining equation of this case can be reduced to $c_s = 1$

This gives a quintuple algebra which may be called (ag_s) of which the multiplication table is

Quintuple algebra

ag_s	i	j	k	l	m
i	j	k	0	0	l
j	k	0	0	0	0
k	0	0	0	0	0
l	0	0	0	0	k
m	l	0	0	k	$j+k$

$[2312^21^22^2]$. The defining equation of this case is

$$c_s = 0$$

This gives a quintuple algebra which may be called (ak_s) of which the multiplication table is

ak_s	i	j	k	l	m
i	j	k	0	0	l
j	k	0	0	0	0
k	0	0	0	0	0
l	0	0	0	0	k
m	l	0	0	k	j

$[2312^212]$. The defining equation of this case is

$$d_{s1} = -1$$

which gives

$$c_{s4} = -1, \quad ml = -k$$

There are two cases;

$[2312^2121]$. When c_{s1} does not vanish

$[2312^212^2]$. When c_{s1} vanishes.

$[2312^212^1]$. The defining equation of this case can be reduced to $c_{51} = 1$

This gives a quintuple algebra which may be called $(a_i)_5$ of which the multiplication table is

a_i	i	j	k	l	m
i	j	k	0	0	l
j	k	0	0	0	0
k	0	0	0	0	0
l	0	0	0	0	k
m	$k-l$	0	0	$-k$	$j+ak$

$[2312^212^2]$ The defining equation of this case is $mi = -l$

There are two cases;

$[2312^212^21]$. When c_5 does not vanish,

$[2312^212^2]$. When c_5 vanishes.

$[2312^212^21]$. The defining equation of this case can be reduced to

$$c_5 = 1$$

This gives a quintuple algebra which may be called $(a_j)_5$ of which the multiplication table is

Quintuple algebra

a_5	i	j	k	l	m
i	j	k	0	0	l
j	k	0	0	0	0
k	0	0	0	0	0
l	0	0	0	0	k
m	l	0	0	k	$j+k$

$[2312^21^22^2]$. The defining equation of this case is

$$c_5 = 0$$

This gives a quintuple algebra which may be (a_5) of which the multiplication table is

a_5	i	j	k	l	m
i	j	k	0	0	l
j	k	0	0	0	
k	0	0	0		
l	0	0	0		
m	l	0			

$[2312^21^22]$. The defining

which gives

There are two

k

j for m , and
to make

risk

equation of this case can

k ,

ple algebra which may be
which the multiplication table is

al	i	j	k	l	m
i	j	k	o	o	l
j	k	o	o	o	o
k	o	o	o	o	o
l	o	o	o	o	o
m	al	o	o	o	k

Quintuple algebra

a_{j_5}	i	j	k	l	m
i	j	k	0	0	l
j	k	0	0	0	0
k	0	0	0	0	0
l	0	0	0	0	k
m	$-l$	0	0	$-k$	$j+k$

$[2312^212^3]$. The defining equation of this case is $m^2 = j$

This gives a quintuple algebra which may be called (ak_5) of which the multiplication table is

ak_5	i	j	k	l	m
i	j	k	0	0	l
j	k	0	0	0	0
k	0	0	0	0	0
l	0	0	0	0	k
m	$-l$	0	0	$-k$	j

$[2312^3]$. The defining equations of this case are $ml = lm = 0, \quad m^2 = c_5 k$

There are two cases;

$[2312^31]$. When d_{s1} is not unity,

$[2312^4]$. When d_{s1} is unity.

$[2312^31]$. The defining equation of this case is $d_{s1} \neq 1$

which gives

Quintuple algebra

$$i [(1-d_{s_1}) m - c_{s_1} j] = (1-d_{s_1}) l - c_{s_1} k$$

$$i [(1-d_{s_1}) l - c_{s_1} k] = 0$$

$$[(1-d_{s_1}) l - c_{s_1} k] i = 0$$

$$[(1-d_{s_1}) m - c_{s_1} j] i = d_{s_1} [(1-d_{s_1}) l - c_{s_1} k]$$

$$[(1-d_{s_1}) l - c_{s_1} k] [(1-d_{s_1}) m - c_{s_1} j] = 0$$

$$[(1-d_{s_1}) m - c_{s_1} j] [(1-d_{s_1}) l - c_{s_1} k] = 0$$

$$[(1-d_{s_1}) m - c_{s_1} j]^2 = (1-d_{s_1})^2 c_s k$$

so that the substitution of $(1-d_{s_1}) m - c_{s_1} j$ for m , and of $(1-d_{s_1}) l - c_{s_1} k$ for l , is the same as to make

$$c_{s_1} = 0$$

and there are two cases;

$[2312^31^2]$. When c_s does not vanish

$[2312^312]$. When c_s vanishes.

$[2312^31^2]$. The defining equation of this case can be reduced to $m^2 = k$,

This gives a quintuple algebra which may be called $(a l_s)$ of which the multiplication table is

$a l_s$	i	j	k	l	m
i	j	k	0	0	l
j	k	0	0	0	0
k	0	0	0	0	0
l	0	0	0	0	0
m	al	0	0	0	k

Quintuple algebra

$[2312^312]$. The defining equation of this case is

$$m^2 = 0$$

This gives a quintuple algebra which may be called (am_s) of which the multiplication table is

am_s	i	j	k	l	m
i	j	k	0	0	l
j	k	0	0	0	0
k	0	0	0	0	0
l	0	0	0	0	0
m	al	0	0	0	0

$[2312^4]$. The defining equation of this case is

$$d_{s1} = 1$$

There are two cases;

$[2312^41]$. When c_{s1} does not vanish

$[2312^5]$. When c_{s1} vanishes.

$[2312^41]$. The defining equation of this case is easily reduced to

$$c_{s1} = 1$$

There are two cases;

$[2312^41^2]$. When c_s does not vanish,

$[2312^412]$. When c_s vanishes.

$[2312^41^2]$. The defining equation of this case is easily reduced to

$$m^2 = k$$

This gives a quintuple algebra which may be called (an_s) of which the multiplication table is

Quintuple algebra

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a_{π_s}	i	j	k	l	m
i	j	k	0	0	l
j	k	0	0	0	0
k	0	0	0	0	0
l	0	0	0	0	0
m	$l+k$	0	0	0	k

$[23/2^4/12]$. The defining equation of this case is
 $m^2 = 0$

This gives a quintuple algebra which may be called (a_0) of which the multiplication table is

a_0	i	j	k	l	m
i	j	k	0	0	l
j	k	0	0	0	0
k	0	0	0	0	0
l	0	0	0	0	0
m	$l+k$	0	0	0	0

$[23/2^5]$. The defining equation of this case is
 $mi = l$

There are two cases;

$[23/2^5/1]$. When c_s does not vanish

$[23/2^5]$. When c_s vanishes.

$[23/2^5/1]$. The defining equation of this case can be reduced to

$$m^2 = k$$

Quintuple algebra

This gives a quintuple algebra which may be called (a_{p_5}) of which the multiplication table is

a_{p_5}	i	j	k	l	m
i	j	k	0	0	l
j	k	0	0	0	0
k	0	0	0	0	0
l	0	0	0	0	0
m	l	0	0	0	k

[2312⁶]. The defining equation of this case is

$$m^2 = 0$$

This gives a quintuple algebra which may be called (a_{q_5}) of which the multiplication table is

a_{q_5}	i	j	k	l	m
i	j	k	0	0	l
j	k	0	0	0	0
k	0	0	0	0	0
l	0	0	0	0	0
m	l	0	0	0	0

[232]. The defining equation of this case is

$$im = 0$$

which gives

$$0 = jm = km$$

and it may be assumed that

$$li = 0$$

which gives

$$lj = lk = 0 = il^2 = l^2i = ilm = iml = mli = im$$

There are two cases;

$$[2321]. \text{ When } mi = l$$

$$[232^2]. \text{ When } mi = 0$$

$[2321]$. The defining equation of this case is
 $mi = l$

which gives

$$0 = mj = mk$$

$$lm = c_{45}k + d_{45}l + c_{45}m$$

$$lmi = l^2 = c_{45}l, \quad 0 = l^4 = c_{45}l^3 = c_{45} = l^2$$

$$m^2 = c_5k + d_5l = c_5m$$

$$m^2i = ml = c_5l, \quad 0 = m^4l = c_5m^3l = c_5 = ml$$

$$0 = lm^2 = d_{45}lm = d_{45}$$

$$0 = m^3 = d_5lm = d_5c_{45}$$

There are two cases;

$$[2321^2]. \text{ When } c_{45} \text{ does not vanish,}$$

$$[2321^2]. \text{ When } c_{45} \text{ vanishes}$$

$[2321^2]$. The defining equation of this case can be reduced to
 $lm = k$

which gives

$$m^2 = c_5k, \quad (m - c_5l)^2 = 0$$

so that the substitution of $m - c_5l$ for m is the same as to make
 $c_5 = 0$

Quintuple algebra

This gives a quintuple algebra which may be called $(a r_5)$ of which the multiplication table is

$a r_5$	i	j	k	l	m
i	j	k	0	0	0
j	k	0	0	0	0
k	0	0	0	0	0
l	0	0	0	0	k
m	l	0	0	0	0

$[23212]$. The defining equation of this case is
 $l m = 0$

There are two cases;

$[232121]$. When d_5 does not vanish,

$[23212^2]$. When d_5 vanishes,

$[232121]$. The defining equation of this case can be reduced to
 $d_5 = 1$

There are two cases;

$[232121^2]$ When c_5 does not vanish,

$[2321212]$. When c_5 vanishes

$[232121^2]$. The defining equation of this case can be reduced to

$$c_5 = 1$$

This gives a quintuple algebra which can be called $(a s_5)$ of which the multiplication table is

Quintuple algebra

///

$a s_s$	i	j	k	l	m
i	j	k	0	0	0
j	k	0	0	0	0
k	0	0	0	0	0
l	0	0	0	0	0
m	l	0	0	0	$k+l$

$[2321212]$. The defining equation of this case is

$$c_s = 0$$

This gives a quintuple algebra which may be called $(a t_s)$ of which the multiplication table is

$a t_s$	i	j	k	l	m
i	j	k	0	0	0
j	k	0	0	0	0
k	0	0	0	0	0
l	0	0	0	0	0
m	l	0	0	0	l

$[23212^3]$. The defining equation of this case is

$$m^2 = c_s k$$

There are two cases;

$[23212^21]$. When c_s does not vanish,

$[23212^3]$. When c_s vanishes.

$[23212^21]$. The defining equation of this case can be reduced to

$$m^2 = k$$

This gives a quintuple algebra which may be called $(a u_5)$ of which the multiplication table is

au_5	i	j	k	l	m
i	j	k	0	0	0
j	k	0	0	0	0
k	0	0	0	0	0
l	0	0	0	0	0
m	l	0	0	0	k

$[23212^3]$. The defining equation of this case is
 $m^2 = 0$.

This gives a quintuple algebra which may be called (av_5) of which the multiplication table is

av_5	i	j	k	l	m
i	j	k	0	0	0
j	k	0	0	0	0
k	0	0	0	0	0
l	0	0	0	0	0
m	l	0	0	0	0

$[232^2]$. The defining equation of this case is
 $mi = 0$

which gives

$$0 = mj = mk = lmi = m^2 0$$

and there is no pure algebra in this case

$[24]$. The defining equation of this case is

$$i^3 = 0$$

and by § 59

$$i^2 = j$$

There are three cases;

$$[241] \text{ When } ik = l, \quad il = m$$

$$[242] \text{ When } ik = l, \quad il = im = 0$$

$$[243] \text{ When } ik = il = im = 0$$

[241]. The defining equations of this case are

$$ik = l, \quad il = m$$

which give

$$0 = im = iml = ml^2 = \epsilon_{s_4} ml^2 = \epsilon_{s_4}, \quad ik = m$$

$$il^2 = ml = b_{s_4} j, \quad l^2 = b_{s_4} i + \epsilon_4 j + \epsilon_4 m$$

$$0 = l^3 = b_{s_4} m + \epsilon_4 ml = b_{s_4} = ml$$

$$im^2 = 0, \quad m^2 = b_5 j + \epsilon_5 m, \quad 0 = m^3 = \epsilon_5 m^2 = \epsilon_5$$

$$imi = a, \quad mi = b_{s_1} j + \epsilon_{s_1} m, \quad mj = \epsilon_{s_1} mi, \quad mi^2 = 0 = \epsilon_{s_1}$$

$$ili = mi = b_{s_1} j, \quad li = b_{s_1} i + b_{s_1} j + \epsilon_{s_1} m$$

$$lil = lm = b_{s_1} m, \quad 0 = l^3 m = b_{s_1} = lm = mi = mil = m^2 l j$$

$$ik^2 = lk = a_3 j + \epsilon_3 l + d_3 m$$

$$ilk = mk = \epsilon_3 m, \quad lik = \epsilon^2 = a_{s_1} m$$

$$0 = mk^2 = \epsilon_3^2 m = \epsilon_3 = mk = k^2 m$$

$$kj = ki^2 = a_{s_1} j + d_{s_1} li = a_{s_1} (1 + d_{s_1}) j + d_{s_1}^2 m$$

$$km = kil = km = a_{s_1} (1 + d_{s_1}) m,$$

$$0 = k^2 m = a_{s_1} (1 + d_{s_1}) km = km, \quad kj = d_{s_1}^2 m$$

$$0 = k^3 = a_3 l + b_3 m + d_3 lk = a_3 = b_3 + d_3^2 = b_3 kj + d_3 kl$$

Quintuple algebra

$$kl = a_{31}l + (b_{31} + d_3 d_{31})m, \quad 0 = d_3 a_{31} = b_{31}d_3 + d_3^2 d_{31} + b_3 d_{31}^2$$

$$0 = klk = a_{31}lk = d_3 a_{31} = lkl = a_{31}l^2 = a_{31} = l^2$$

$$0 = ki + iki + i^2 k = (d_{31}^2 + d_{31} + 1)m, \quad d_{31} = \sqrt{-1} = i$$

$$0 = k^2 i + kik + ik^2 = b_{31} + d_3(1 + id_{31})$$

$$i(k + bi) = l + bj, \quad i(l + bi) = m$$

$$(k + bi)i = b_{31}j + d_{31}l + \epsilon_{31}m + \mu j = (b_{31} + \mu - \mu d_{31})j + d_{31}(l + \mu i) + \epsilon_{31}m$$

$$(l + \mu j)i = d_{31}m$$

so that if μ satisfies the equation

$$\mu(d_{31} - 1) = b_{31}$$

the substitution of $k + \mu i$ for k and of $l + \mu j$ for l is the same as to make

$$0 = b_{31} = d_3 = b_3$$

There are four cases;

[241²]. When neither ϵ_{31} nor ϵ_3 vanishes

[241²]. When ϵ_{31} does not vanish but ϵ_3 vanishes

[241³]. When ϵ_{31} vanishes and not ϵ_3

[241⁴]. When ϵ_{31} and ϵ_3 both vanish

[241²]. The defining equations of this case can be reduced without loss of generality to

$$\epsilon_{31} = \epsilon_3 = 1$$

This gives a quintuple algebra which may be called (aw_5) of which the multiplication table is

Quintuple algebra

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ax_s	i	j	k	l	m
i	j	0	l	m	0
j	0	0	m	0	0
k	$rl+m$	r^2m	m	0	0
l	rm	0	0	0	0
m	0	0	0	0	0

[2412]. The defining equations of this case can be reduced to

$$\varepsilon_{31} = 1, \quad \varepsilon_3 = 0$$

This gives a quintuple algebra which may be called (ax_s) of which the multiplication table is

ax_s	i	j	k	l	m
i	j	0	l	m	0
j	0	0	m	0	0
k	$rl+m$	r^2m	0	0	0
l	rm	0	0	0	0
m	0	0	0	0	0

[2413]. The defining equations of this case can be reduced to

$$\varepsilon_{31} = 0, \quad \varepsilon_3 = 1$$

This gives a quintuple algebra which may be called (ay_s) of which the multiplication table is

Quintuple algebra

a_7	i	j	k	l	m
i	j	0	l	m	0
j	0	0	m	0	0
k	rl	rm	m	0	0
l	rm	0	0	0	0
m	0	0	0	0	0

[2414]. The defining equations of this case are

$$\varepsilon_{31} = \varepsilon_3 = 0$$

This gives a quintuple algebra which may be called (a_7) of which the multiplication table is

a_7	i	j	k	l	m
i	j	0	l	m	0
j	0	0	m	0	0
k	rl	rm	0	0	0
l	rm	0	0	0	0
m	0	0	0	0	0

[242]. The defining equations of this case are

$$ik = l, \quad il = im = 0$$

which give

$$\begin{aligned} 0 &= jk = jl = jm \\ li &= iki = a_{31}j + c_{31}l \\ 0 &= l^2i = c_{31} = li^2 = lj \\ 0 &= a_{31}jk = lik = l^2 = ikl = a_{34} = c_{34} \\ ik^2 &= lk = a_{31}j + c_{31}l \end{aligned}$$

Quintuple algebra

$$0 = lk^2 = e_3 = imi = a_{s1} = e_{s1}$$

$$mi^2 = mj = d_{s1} li + e_{s1} mi$$

$$0 = mji = e_{s1}$$

$$mj = a_{s1} d_{s1} j$$

$$0 = m^2 j = a_{s1} d_{s1} = mj$$

$$ikm = lm = a_{s5} j + e_{s5} l$$

$$0 = lm^2 = e_{s5}$$

$$0 = i^2 k + iki + ki^2 = e_{s1} d_{s1} = 2a_{s1} + a_{s1} d_{s1} + e_{s1} b_{s1}$$

$$kj = -a_{s1} j, \quad 0 = k^2 j = a_{s1} = kj = li = e_{s1} b_{s1}$$

$$0 = imk = a_{s3} = e_{s3}$$

$$0 = mk^2 = e_{s3}$$

$$mik = ml = a_3 d_{s1} j$$

$$kik = kl = (a_3 d_{s1} + e_{s1} b_{s2}) j + e_{s1} d_{s2} l$$

$$0 = k^2 l = e_{s1} d_{s2} = lki = e_{s1} lm = e_{s1} a_{s5} = k^2 m = e_{s5}$$

There are two cases;

[2421]. When e_{s1} does not vanish,

[2422]. When e_{s1} vanishes.

[2421]. The defining equation of this case can be reduced to

$$ki = m$$

which gives

$$0 = mi = kil = ml = kim = m^2$$

$$a_3 j = ik^2 = k^2 i = lk = km, \quad 0 = lm$$

$$b_{s2} j = kik = kl = mk$$

$$0 = ik^2 + kik + k^2 i = 2a_3 + b_{s2}$$

Quintuple algebra

$$0 = k^3 = a_3 = b_{33} = k\ell = km = mk = ml$$

and if μ is determined by the equation

$$\mu^2 + (d_3 + \varepsilon_3) \mu - b_3 = 0$$

$k + \mu i$, $\ell + \mu j$, and $m + \mu j$ can be respectively substituted for k , ℓ and m which is the same as to make

$$b_3 = 0$$

There are three cases;

$[2421^3]$. When neither d_3 nor ε_3 vanishes,

$[24212]$. When d_3 vanishes and not ε_3 ,

$[24213]$. When d_3 and ε_3 both vanish.

$[2421^3]$. The defining equation of this case can be reduced to

$$d_3 = 1$$

This gives a quintuple algebra which may be called (ba_5) of which the multiplication table is

ba_5	i	j	k	ℓ	m
i	j	0	ℓ	0	0
j	0	0	0	0	0
k	m	0	$\ell + am$	0	0
ℓ	0	0	0	0	0
m	0	0	0	0	0

$[2421^3]$. The defining equation of this case can be reduced to

$$k^2 = m$$

Quintuple algebra

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This gives a quintuple algebra which may be called (bb_5) of which the multiplication table is

bb_5	i	j	k	l	m
i	j	0	l	0	0
j	0	0	0	0	0
k	m	0	m	0	0
l	0	0	0	0	0
m	0	0	0	0	0

[24213]. The defining equation of this case is
 $k^2 = 0$

This gives a quintuple algebra which may be called (bc_5) of which the multiplication table is

bc_5	i	j	k	l	m
i	j	0	l	0	0
j	0	0	0	0	0
k	m	0	0	0	0
l	0	0	0	0	0
m	0	0	0	0	0

[242] The defining equation of this case is
 $\varepsilon_{31} = 0$

There are two cases;

[242¹]. When ε_3 does not vanish

[242²]. When ε_3 vanishes

[242¹]. The defining equation of this case can be

Quintuple algebra

reduced to
which gives

$$k^2 = a_3 i + m$$

$$k i k = k l = a_3 d_{31} j$$

$$i k^2 = l k = a_3 j$$

$$k^2 i = a_3 j + m i = d_{31} k l = a_3 d_{31}^2 j$$

$$0 = k^2 i + i k + k i k = a_3 (d_{31}^2 + d_{31} + 1) j$$

$$m i = a_3 (d_{31} l - 1) j, \quad m l = m i k = 0$$

$$0 = k^3 = a_3 l + m k = a_3 k i + k m$$

$$m k = -a_3 l, \quad k m = -a_3 b_{31} j - a_3 d_{31} l, \quad l m = 0$$

There are two cases;

[242²1²] When a_3 does not vanish,

[242²1²] When a_3 vanishes,

[242²1²]. The defining equation of this case can be reduced to

$$k^2 = i + m$$

which gives

$$d_{31} = \dot{V}_1 = r, \quad l k = j, \quad m k = -l,$$

$$k i = -k m = b_{31} j + r l, \quad m i = (r^2 - 1) j, \quad m^2 = -r j$$

There are two cases:

[242²1³]. When b_{31} does not vanish,

[242²1³2]. When b_{31} vanishes.

[242²1³]. The defining equation of this case can be reduced to

$$k i = j + r l.$$

Quintuple algebra

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This gives a quintuple algebra which may be called $(b d_s)$ of which the multiplication table is

$b d_s$	i	j	k	l	m
i	j	0	l	0	0
j	0	0	0	0	0
k	$j+rl$	0	$i+m$	rl	$-j-rl$
l	0	0	j	0	0
m	$(r-1)j$	0	$-l$	0	$-rj$

$[242^2 1^2 2]$. The defining equation of this case is
 $ki = rl$

This gives a quintuple algebra which may be called $(b e_s)$ of which the multiplication table is

$b e_s$	i	j	k	l	m
i	j	0	l	0	0
j	0	0	0	0	0
k	rl	0	$i+m$	rl	$-rl$
l	0	0	j	0	0
m	$(r-1)j$	0	$-l$	0	$-rj$

$[242^2 1^2 2]$. The defining equation of this case is
 $k^2 = m$

which gives

$$0 = kl = lk = km = mk = m^2 = k^2 i = mi$$

There are two cases;

$[242^2 1^2 2]$. When b_{s1} does not vanish

Quintuple algebra

$[242^2 12^2]$ When b_3 vanishes

$[242^2 12^1]$ The defining equation of this case can be reduced to

$$ki = j + d_3 l$$

This gives a quintuple algebra which may be called (bf_5) of which the multiplication table is

bf_5	i	j	k	l	m
i	j	0	l	0	0
j	0	0	0	0	0
k	$j + al$	0	m	0	0
l	0	0	0	0	0
m	0	0	0	0	0

$[242^2 12^2]$. The defining equation of this case is

$$ki = d_3 l$$

This gives a quintuple algebra which may be called (bg_5) of which the multiplication table is

bg_5	i	j	k	l	m
i	j	0	l	0	0
j	0	0	0	0	0
k	al	0	m	0	0
l	0	0	0	0	0
m	0	0	0	0	0

$[242^1]$. The defining equation of this case is

$$\epsilon_3 = 0$$

Quintuple algebra

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which gives

$$0 = k^2 i = a_3, j = a_3 = \ell k = m \ell = k \ell = m^2 = \varepsilon_5 = d_5 a_{35} = k^2 m = \varepsilon_{35}$$

$$0 = k m k = a_{35} \ell = a_{35} = \ell m$$

There are two cases

$[242^3 1]$ When d_5 does not vanish

$[242^4]$. When d_5 vanishes.

$[242^3 1]$. The defining equation of this case can be reduced to

$$d_5 = 1$$

which gives

$$i(k + b_5 i + \mu m) = \ell + b_5 j = m^2$$

and if

$$\mu^2 = -d_3 - b_5 - b_5 d_{31} - b_5 \mu d_{51} - \mu d_{35} - \mu d_{53}$$

the substitution of $k + b_5 i + \mu m$ for k and $\ell + b_5 j$ for ℓ is the same as to make

$$b_5 = d_3 = 0$$

which gives

$$m^2 = \ell, \quad k^2 = b_5 j$$

There are two cases;

• $[242^3 1^2]$ When b_5 does not vanish,

$[242^3 1^2]$. When b_5 vanishes.

$[242^3 1^2]$. The defining equation of this case can be reduced to

$$k^2 = j$$

This gives a quintuple algebra which can be called (bh_5) of which the multiplication table is

Quintuple algebra

b_{31}	i	j	k	l	m
i	j	0	l	0	0
j	0	0	0	0	0
k	$aj+bl$	0	j	0	$cj+d\ell$
l	0	0	0	0	0
m	$a'j+b'l$	0	$c'j+d'l$	0	l

$[242^312]$ The defining equation of this case is
 $k^2 = 0$

There are two cases

$[242^3121]$, When b_{31} does not vanish

$[242^312^2]$, When b_{31} vanishes.

$[242^3121]$ The defining equation of this case can be reduced to
 $b_{31} = 1$

This gives a quintuple algebra which may be called (b_{31}) of which the multiplication table is

b_{31}	i	j	k	l	m
i	j	0	l	0	0
j	0	0	0	0	0
k	$j+al$	0	0	0	$bj+cl$
l	0	0	0	0	0
m	$a'j+b'l$	0	$c'j+d'l$	0	l

$[242^312^2]$. The defining equation of this case is
 $ki = d_{31}l$

There are two cases;

Quintuple algebra

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$[242^3 12^2 1]$. When b_{s_1} does not vanish

$[242^3 12^3]$ When b_{s_1} vanishes.

$[242^3 12^2 1]$. The defining equation of this case can be reduced to $b_{s_1} = 1$

This gives a quintuple algebra which may be called (b_{j_s}) of which the multiplication table is

b_{j_s}	i	j	k	l	m
i	j	0	l	0	0
j	0	0	0	0	0
k	al	0	0	0	$b_j + cl$
l	0	0	0	0	0
m	$j + al$	0	$b_j + il$	0	l

$[242^3 12^3]$ The defining equation of this case is

$$mi = d_{s_1} l$$

which can always, in the case of a single algebra, be reduced to $mi = l$

This gives a quintuple algebra which may be called (b_{k_s}) of which the multiplication table is

b_{k_s}	i	j	k	l	m
i	j	0	l	0	0
j	0	0	0	0	0
k	al	0	0	0	$b_j + cl$
l	0	0	0	0	0
m	l	0	$aj + bl$	0	l

Quintuple algebra

b_{31}	i	j	k	l	m
i	j	0	l	0	0
j	0	0	0	0	0
k	$a_j + b_l$	0	j	0	$c_j + d_l$
l	0	0	0	0	0
m	$a_j + b_l$	0	$c_j + d_l$	0	l

$[242^312]$ The defining equation of this case is
 $k^2 = 0$

There are two cases

$[242^3121]$, When b_{31} does not vanish

$[242^312^2]$, When b_{31} vanishes.

$[242^3121]$ The defining equation of this case can be reduced to
 $b_{31} = 1$

This gives a quintuple algebra which may be called (b_{31}) of which the multiplication table is

b_{31}	i	j	k	l	m
i	j	0	l	0	0
j	0	0	0	0	0
k	$j + a_l$	0	0	0	$b_j + c_l$
l	0	0	0	0	0
m	$a_j + b_l$	0	$c_j + d_l$	0	l

$[242^312^2]$. The defining equation of this
 $ki = d_{31} l$

There are two cases;

Quintuple

$[242^312^21]$. When i is even

$[242^312^3]$ When i is odd

$[242^312^21]$. The defining equations are reduced to

This gives a quintuple system called (b_j) of which the

b_j	i	j	k	l
i	i	j	k	l
j	0	i	k	l
k	al	i	j	l
l	0	i	j	k
m	$j+al$	i	k	l

$[242^312^3]$ The defining equations are

$$m_i = 2, 1$$

which can always be reduced to

This gives a quintuple system called (b_k) of which the

vanish,

es,

if this case can be

algebra which may be multiplication table is

Quintuple algebra

b_{n_5}	i	j	k	l	m
i	j	0	l	0	0
j	0	0	0	0	0
k	$-l$	0	0	0	al
l	0	0	0	0	0
m	j	0	$j-al$	0	ej

[242]. The defining equation of this case is

$$b_{s_3} = 0$$

This gives a quintuple algebra which may be called (b_{o_5}) of which the multiplication table is

b_{o_5}	i	j	k	l	m
i	j	0	l	0	0
j	0	0	0	0	0
k	$-l$	0	0	0	al
l	0	0	0	0	0
m	j	0	$-al$	0	ej

[243]. The defining equations of this case are

$$0 = ik = il = im$$

which give $0 = jk = jl = jm$

There are two cases;

[2431] When $ki = l$, $li = m$, $mi = 0$

[2432] When $ki = l$, $li = mi = 0$

[2431]. The defining equations of this case are

$$ki = l, \quad li = m, \quad mi = 0$$

which give

$$kj = m \quad \ell j = mj = 0 = \ell k = mk = \ell^2 = \ell m = m\ell = m^2$$

$$0 = ik^2 = ikl = ikm = a_3 = a_{34} = a_{35}$$

$$k'i = k\ell = c_3 \ell = d_3 m$$

$$k\ell i = km = c_3 m, \quad 0 = k^3 m = c_3 = km$$

$$0 = k^3 = b_3 m + d_3^2 m = b_3 + d_3^2$$

There are two cases

$[2431^2]$ When ϵ_3 does not vanish

$[24312]$. When ϵ_3 vanishes

$[2431^2]$. The defining equation of this case can be reduced to

$$\epsilon_3 = 1$$

This gives a quintuple algebra which may be called (b/p_5) of which the multiplication table is

b/p_5	i	j	k	ℓ	m
i	j	0	0	0	0
j	0	0	0	0	0
k	ℓ	m	$-a_j + a_l + m$	am	0
ℓ	m	0	0	0	0
m	0	0	0	0	0

$[24312]$. The defining equation of this case is

$$\epsilon_3 = 0$$

This gives a quintuple algebra which may be called (b/q_5) of which the multiplication table is

Quintuple algebra

b_{q_5}	i	j	k	l	m
i	j	0	0	0	0
j	0	0	0	0	0
k	l	m	$-aj+al$	am	0
l	m	0	0	0	0
m	0	0	0	0	0

[2432] The defining equations of this case are
 $ki = \epsilon$, $li = mi = 0$

which give

$$0 = kj = lj = mj = lk = l^2 = lm = ik^2 = a_3$$

$$kl = c_3 l, \quad 0 = k^3 l = c_3 = kl$$

$$0 = ikm = a_{35} = kmi = c_{35} = k^3 m = \epsilon_{35} = imk = a_{53}$$

$$ml = c_{53} l, \quad 0 = m^3 l = c_{53} = ml = mk^2 = \epsilon_{53}$$

and it may be assumed that

$$k^2 = m$$

which gives $0 = k^3 = km = mk = m^2$

There is then a quintuple algebra which may be called (b_{τ_5}) of which the multiplication table is

b_{τ_5}	i	j	k	l	m
i	j	0	0	0	0
j	0	0	0	0	0
k	l	0	m	0	0
l	0	0	0	0	0
m	0	0	0	0	0

Quintuple algebra

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[25]. The defining equations of this case are

$$0 = i^2 = j^2 = k^2 = l^2 = m^2 = ij + ji = ik + ki = il + li = im + mi$$

$$0 = jk + kj = jl + lj = jm + mj = kl + lk = km + mk = lm + ml$$

and it may be assumed that

$$ij = k = -ji, \quad il = m = -li$$

which gives

$$0 = ik = ki = jk = kj = im = mi = km = mk = lm = ml$$

$$ijk = kl = b_{24}k + d_{24}m = -ilj = -mj = jm$$

$$0 = j^2m = d_{24}jm = d_{24}kl = b_{24}kl = b_{24}kl = lk = jm = mj$$

$$0 = j^2l = a_{24}k = a_{24}$$

$$i(c_{24}j + e_{24}l) = c_{24}k + e_{24}m$$

$$j(c_{24}j + e_{24}k) = e_{24}(c_{24}k + e_{24}m)$$

$$l(c_{24}j + e_{24}l) = -c_{24}(c_{24}k + e_{24}m)$$

so that it is easy to see that there is no pure algebra in this case

Sextuple algebra

Sextuple Algebra

There are two cases;

- [1]. When there is an idempotent basis
- [2]. When the algebra is nilpotent
- [1] The defining equation of this case is

$$i^2 = i$$

There are 19 cases;

- [1']. When all the other units but i are in the first group.
- [12]. When j, k, l, m and n are in the first and n in the second group,
- [13]. When j, k and l are in the first and m and n in the second group,
- [14] When j, k and l are in the first, m in the second and n in the third group,
- [15]. When j and k are in the first and l, m and n in the second group,
- [16]. When j and k are in the first, l and m in the second and n in the third group,
- [17]. When j and k are in the first, l the second, m the third and n the fourth group,
- [18]. When j is in the first, and k, l, m and n in the

second group.

[19]. When j is in the first, k l and m in the second and n in the third group,

[10'] When j is in the first, k and l the second, and m and n in the third group

[11']. When j is in the first, k and l the second, m the third and n the fourth group

[12']. When j is in the first k the second, l the third and m and n the fourth group

[13']. When j, k, l, m and n are in the second group.

[14']. When j, k, l and m are in the second and n the third group

[15']. When j, k and l are in the second and m and n are in the third group

[16']. When j, k and l are in the second, m the third and n the fourth group

[17']. When j and k are in the second, l and m the third and n the fourth group

[18']. When j and k are in the second, l the third and m and n the fourth group

[19']. When j is in the second, k the third and l, m and n the fourth group

[1']. The defining equations of this case are

$$ij = ji = j, ik = ki = k, il = li = l, im = mi = m, in = ni = n$$

Sextuple algebra

And the 54 algebras of this case deduced from (q_5) to (br_5) may be called (a_6) to (bb_6) and their multiplication tables are the following

a_6	i	j	k	l	m	n
i	i	j	k	l	m	n
j	j	k	l	m	n	0
k	k	l	m	n	0	0
l	l	m	n	0	0	0
m	m	n	0	0	0	0
n	n	0	0	0	0	0

b_6	i	j	k	l	m	n
i	i	j	k	l	m	n
j	j	k	l	m	0	0
k	k	l	m	0	0	0
l	l	m	0	0	0	0
m	m	0	0	0	0	0
n	n	m	0	0	0	m

Sextuple algebra

c_0	i	j	k	l	m	n
i	k	l	k	l	m	n
j	j	k	l	m	0	0
k	k	l	m	0	0	0
l	l	m	0	0	0	0
m	m	0	0	0	0	0
n	n	m	0	0	0	0

d_0	i	j	k	l	m	n
i	i	j	k	l	m	n
j	j	k	l	0	0	m
k	k	l	0	0	0	0
l	l	0	0	0	0	0
m	m	l	0	0	0	l
n	n	$k+m$	$(1+r)l$	0	rl	$k+l(2r-1)m$

e_0	i	j	k	l	m	n
i	i	j	k	l	m	n
j	j	k	l	0	0	m
k	k	l	0	0	0	0
l	l	0	0	0	0	0
m	m	l	0	0	0	l
n	n	$k+m$	$(1+r)l$	0	rl	$k+l(2r-1)m$

f_0	i	j	k	l	m	n
i	i	j	k	l	m	n
j	j	k	l	0	0	m
k	k	l	0	0	0	0
l	l	0	0	0	0	0
m	m	l	0	0	0	l
n	n	$k+m$	0	0	rl	$k+l$

g_0	i	j	k	l	m	n
i	i	j	k	l	m	n
j	j	k	l	0	0	m
k	k	l	0	0	0	0
l	l	0	0	0	0	0
m	m	l	0	0	0	l
n	n	$k+m$	0	0	l	k

h_0	i	j	k	l	m	n
i	i	j	k	l	m	n
j	j	k	l	0	0	0
k	k	l	0	0	0	0
l	l	0	0	0	0	0
m	m	l	0	0	0	0
n	n	$k+m$	$(1+r)l$	0	0	$k+l$

i_0	i	j	k	l	m	n
i	i	j	k	l	m	n
j	j	k	l	0	0	0
k	k	l	0	0	0	0
l	l	0	0	0	0	0
m	m	l	0	0	0	0
n	n	$k+m$	$(1+r)l$	0	0	$(1+r)m$

j_0	i	j	k	l	m	n
i	i	j	k	l	m	n
j	j	k	l	0	0	m
k	k	l	0	0	0	0
l	l	0	0	0	0	0
m	m	l	0	0	0	0
n	n	$k+m$	0	0	0	l

k_0	i	j	k	l	m	n
i	i	j	k	l	m	n
j	j	k	l	0	0	m
k	k	l	0	0	0	0
l	l	0	0	0	0	0
m	m	l	0	0	0	0
n	n	$k+m$	0	0	0	0

Sextuple algebra

l_6	i	j	k	l	m	n
i	i	j	k	l	m	n
j	j	k	l	o	o	m
k	k	l	o	o	o	o
l	l	o	o	o	o	o
m	m	o	o	o	o	l
n	n	$l+m$	o	o	l	$k+m$

m_6	i	j	k	l	m	n
i	i	j	k	l	m	n
j	j	k	l	o	o	m
k	k	l	o	o	o	o
l	l	o	o	o	o	o
m	m	o	o	o	o	l
n	n	m	o	o	l	$k+m$

n_6	i	j	k	l	m	n
i	i	j	k	l	m	n
j	j	k	l	o	o	m
k	k	l	o	o	o	o
l	l	o	o	o	o	o
m	m	o	o	o	o	o
n	n	$l+m$	o	o	o	m

o_6	i	j	k	l	m	n
i	i	j	k	l	m	n
j	j	k	l	o	o	m
k	k	l	o	o	o	o
l	l	o	o	o	o	o
m	m	o	o	o	o	o
n	n	am	o	o	o	m

k_6	i	j	k	l	m	n
i	i	j	k	l	m	n
j	j	k	l	o	o	m
k	k	l	o	o	o	o
l	l	o	o	o	o	o
m	m	o	o	o	o	l
n	n	$l+m$	o	o	l	$k+l$

g_6	i	j	k	l	m	n
i	i	j	k	l	m	n
j	j	k	l	o	o	m
k	k	l	o	o	o	o
l	l	o	o	o	o	o
m	m	o	o	o	o	l
n	n	m	o	o	l	$k+l$

r_6	i	j	k	l	m	n
i	i	j	k	l	m	n
j	j	k	l	o	o	m
k	k	l	o	o	o	o
l	l	o	o	o	o	o
m	m	o	o	o	o	m
n	n	m	o	o	l	k

s_6	i	j	k	l	m	n
i	i	j	k	l	m	n
j	j	k	l	o	o	m
k	k	l	o	o	o	o
l	l	o	o	o	o	o
m	m	o	o	o	o	l
n	n	$l+m$	o	o	$-l$	$k+l$

t_6	i	j	k	l	m	n
i	i	j	k	l	m	n
j	j	k	l	o	o	m
k	k	l	o	o	o	o
l	l	o	o	o	o	o
m	m	o	o	o	o	l
n	n	$-m$	o	o	$-l$	$k+l$

Sextuple algebra

u_6	i	j	k	l	m	n
i	i	j	k	l	m	n
j	j	k	l	o	o	m
k	k	l	o	o	o	o
l	l	o	o	o	o	o
m	m	o	o	o	o	l
n	n	m	o	o	l	k

v_6	i	j	k	l	m	n
i	i	j	k	l	m	n
j	j	k	l	o	o	m
k	k	l	o	o	o	o
l	l	o	o	o	o	o
m	m	o	o	o	o	o
n	n	am	o	o	o	l

w_6	i	j	k	l	m	n
i	i	j	k	l	m	n
j	j	k	l	o	o	n
k	k	l	o	o	o	o
l	l	o	o	o	o	o
m	m	o	o	o	o	o
n	n	am	o	o	o	o

x_6	i	j	k	l	m	n
i	i	j	k	l	m	n
j	j	k	l	o	o	m
k	k	l	o	o	o	o
l	l	o	o	o	o	o
m	m	o	o	o	o	o
n	n	$l+m$	o	o	o	l

y_6	i	j	k	l	m	n
i	i	j	k	l	m	n
j	j	k	l	o	o	o
k	k	l	o	o	o	o
l	l	o	o	o	o	o
m	m	o	o	o	o	o
n	n	$l+m$	o	o	o	o

z_6	i	j	k	l	m	n
i	i	j	k	l	m	n
j	j	k	l	o	am	m
k	k	l	o	o	o	o
l	l	o	o	o	o	o
m	m	o	o	o	o	o
n	n	m	o	o	o	l

aa_6	i	j	k	l	m	n
i	i	j	k	l	m	n
j	j	k	l	o	o	m
k	k	l	o	o	o	o
l	l	o	o	o	o	o
m	m	o	o	o	o	o
n	n	m	o	o	o	o

ab_6	i	j	k	l	m	n
i	i	j	k	l	m	n
j	j	k	l	o	o	o
k	k	l	o	o	o	o
l	l	o	o	o	o	o
m	m	o	o	o	o	l
n	n	m	o	o	o	o

ac_6	i	j	k	l	m	n
i	i	j	k	l	m	n
j	j	k	l	o	o	o
k	k	l	o	o	o	o
l	l	o	o	o	o	o
m	m	o	o	o	o	o
n	n	m	o	o	o	$l+m$

Sextuple algebra

a	i	j	k	l	m	n
i	i	j	k	l	m	n
j	j	k	l	0	0	0
k	k	l	0	0	0	0
l	l	0	0	0	0	0
m	m	0	0	0	0	0
n	n	m	0	0	0	m

a	i	j	k	l	m	n
i	i	j	k	l	m	n
j	j	k	l	0	0	0
k	k	l	0	0	0	0
l	l	0	0	0	0	0
m	m	0	0	0	0	0
n	n	m	0	0	0	l

a	i	j	k	l	m	n
i	i	j	k	l	m	n
j	j	k	l	0	0	0
k	k	l	0	0	0	0
l	l	0	0	0	0	0
m	m	0	0	0	0	0
n	n	m	0	0	0	0

a	i	j	k	l	m	n
i	i	j	k	l	m	n
j	j	k	0	m	n	0
k	k	0	0	n	0	0
l	l	m	n	0	0	0
m	m	n	0	0	0	0
n	n	0	0	0	0	0

a	i	j	k	l	m	n
i	i	j	k	l	m	n
j	j	k	0	m	n	0
k	k	0	0	n	0	0
l	l	m	n	0	0	0
m	m	n	0	0	0	0
n	n	0	0	0	0	0

a	i	j	k	l	m	n
i	i	j	k	l	m	n
j	j	k	0	m	n	0
k	k	0	0	n	0	0
l	l	m	n	0	0	0
m	m	n	0	0	0	0
n	n	0	0	0	0	0

a	i	j	k	l	m	n
i	i	j	k	l	m	n
j	j	k	0	m	n	0
k	k	0	0	n	0	0
l	l	m	n	0	0	0
m	m	n	0	0	0	0
n	n	0	0	0	0	0

a	i	j	k	l	m	n
i	i	j	k	l	m	n
j	j	k	0	m	0	0
k	k	0	0	0	0	0
l	l	n	0	m	n	0
m	m	0	0	0	0	0
n	n	0	0	0	0	0

a	i	j	k	l	m	n
i	i	j	k	l	m	n
j	j	k	0	m	0	0
k	k	0	0	0	0	0
l	l	n	0	n	0	0
m	m	0	0	0	0	0
n	n	0	0	0	0	0

Sextuple algebra

a_0	i	j	k	l	m	n
i	i	j	k	l	m	n
j	j	k	0	m	0	0
k	k	0	0	0	0	0
l	l	m	0	0	0	0
m	m	0	0	0	0	0
n	n	0	0	0	0	0

a_0	i	j	k	l	m	n
i	i	j	k	l	m	n
j	j	k	0	m	0	0
k	k	0	0	0	0	0
l	l	krm	0	$j+n$	rk	kpn
m	m	0	0	k	0	0
n	n	$(r-i)k$	0	$-m$	0	$-rk$

a_0	i	j	k	l	m	n
i	i	j	k	l	m	n
j	j	k	0	m	0	0
k	k	0	0	0	0	0
l	l	rm	0	$j+n$	rk	rm
m	m	0	0	k	0	0
n	n	$(r-i)k$	0	$-m$	0	$-rk$

a_0	i	j	k	l	m	n
i	i	j	k	l	m	n
j	j	k	0	m	0	0
k	k	0	0	0	0	0
l	l	$kram$	0	n	0	0
m	m	0	0	0	0	0
n	n	0	0	0	0	0

a_0	i	j	k	l	m	n
i	i	j	k	l	m	n
j	j	k	0	m	0	0
k	k	0	0	0	0	0
l	l	am	0	n	0	0
m	m	0	0	0	0	0
n	n	0	0	0	0	0

a_0	i	j	k	l	m	n
i	i	j	k	l	m	n
j	j	k	0	m	0	0
k	k	0	0	0	0	0
l	l	$aktbm$	0	k	0	$ckdm$
m	m	0	0	0	0	0
n	n	$aktbm$	0	$ckdm$	0	m

a_0	i	j	k	l	m	n
i	i	j	k	l	m	n
j	j	k	0	m	0	0
k	k	0	0	0	0	0
l	l	$kram$	0	0	0	$bkrcm$
m	m	0	0	0	0	0
n	n	$aktbm$	0	$ckdm$	0	m

a_0	i	j	k	l	m	n
i	i	j	k	l	m	n
j	j	k	0	m	0	0
k	k	0	0	0	0	0
l	l	am	0	0	0	$bkrcm$
m	m	0	0	0	0	0
n	n	$kram$	0	$bkrcm$	0	m

a_0	i	j	k	l	m	n
i	i	j	k	l	m	n
j	j	k	0	m	0	0
k	k	0	0	0	0	0
l	l	am	0	0	0	$bkrcm$
m	m	0	0	0	0	0
n	n	m	0	$aktbm$	0	m

Sextuple algebra

$$a_{27} \begin{array}{c|c|c|c|c|c|c} i & j & k & l & m & n \\ \hline i & i & j & k & l & m & n \\ \hline j & j & k & 0 & m & 0 & 0 \\ \hline k & k & 0 & 0 & 0 & 0 & 0 \\ \hline l & l & k-m & 0 & 0 & 0 & ak+bm \\ \hline m & m & 0 & 0 & 0 & 0 & 0 \\ \hline n & n & k & 0 & ak-bl & 0 & ck \end{array}$$

$$a_{28} \begin{array}{c|c|c|c|c|c|c} i & j & k & l & m & n \\ \hline i & i & j & k & l & m & n \\ \hline j & j & k & 0 & m & 0 & 0 \\ \hline k & k & 0 & 0 & 0 & 0 & 0 \\ \hline l & l & -m & 0 & 0 & 0 & k+am \\ \hline m & m & 0 & 0 & 0 & 0 & 0 \\ \hline n & n & k & 0 & bk-am & 0 & ck \end{array}$$

$$a_{29} \begin{array}{c|c|c|c|c|c|c} i & j & k & l & m & n \\ \hline i & i & j & k & l & m & n \\ \hline j & j & k & 0 & m & 0 & 0 \\ \hline k & k & 0 & 0 & 0 & 0 & 0 \\ \hline l & l & -m & 0 & 0 & 0 & am \\ \hline m & m & 0 & 0 & 0 & 0 & 0 \\ \hline n & n & k & 0 & k-am & 0 & ck \end{array}$$

$$a_{30} \begin{array}{c|c|c|c|c|c|c} i & j & k & l & m & n \\ \hline i & i & j & k & l & m & n \\ \hline j & j & k & 0 & m & 0 & 0 \\ \hline k & k & 0 & 0 & 0 & 0 & 0 \\ \hline l & l & -m & 0 & 0 & 0 & am \\ \hline m & m & 0 & 0 & 0 & 0 & 0 \\ \hline n & n & k & 0 & -am & 0 & ck \end{array}$$

$$a_{31} \begin{array}{c|c|c|c|c|c|c} i & j & k & l & m & n \\ \hline i & i & j & k & l & m & n \\ \hline j & j & k & 0 & 0 & 0 & 0 \\ \hline k & k & 0 & 0 & 0 & 0 & 0 \\ \hline l & l & m & n & -ak+am+n & an & 0 \\ \hline m & m & n & 0 & 0 & 0 & 0 \\ \hline n & n & 0 & 0 & 0 & 0 & 0 \end{array}$$

Sextuple algebra

ba_6	i	j	k	l	m	n
i	i	j	k	l	m	n
j	j	k	0	0	0	0
k	k	0	0	0	0	0
l	l	m	n	$-ak+am$	an	0
m	m	n	0	0	0	0
n	n	0	0	0	0	0

bb_6	i	j	k	l	m	n
i	i	j	k	l	m	n
j	j	k	0	0	0	0
k	k	0	0	0	0	0
l	l	m	0	n	0	0
m	m	0	0	0	0	0
n	n	0	0	0	0	0

in all of which r is the imaginary cube root of unity.

[12]. The defining equations of this case are

$$ij = ji = j, ik = ki = k, il = li = l, im = mi = m, in = n, ni = 0$$

which give

$$0 = jn = nj = kn = nk = ln = nl = mn = nm = n^2$$

so that there is no pure algebra in this case

[13]. The defining equations of this case are

$$ij = ji = j, ik = ki = k, il = li = l, im = m, in = n, mi = ni = 0$$

Sextuple algebra

There are four cases, which correspond to relations between the units of the first group such as to form either of the quadratic algebras (a_4) (b_4) (c_4) or (d_4)

[131]. The defining equations of this case are

$j^2 = k$, $jk = kj = l$, $jl = k^2 = kl = lj = lk = l^2 = 0$
and, in the result, we obtain

$$jm = n, \quad jn = km = kn = lm = ln = 0$$

This gives a sextuple algebra which may be called (bc_6) of which the multiplication table is

bc_6	i	j	k	l	m	n
i	i	j	k	l	m	n
j	j	k	l	0	n	0
k	k	l	0	0	0	0
l	l	0	0	0	0	0
m	0	0	0	0	0	0
n	0	0	0	0	0	0

[132]. The defining equations of this case are

$j^2 = k = l$, $lj = ak$, $jk = jl = kj = k^2 = kl = lk = 0$
which give

$$km = kn = 0$$

There are two cases

[1321]. When ϵ_{24} does not vanish

[1322]. When ϵ_{24} vanishes.

[1321]. The defining equation of this case can be reduced to $jn = m$ which gives

$$o = jm = lm$$

This gives a sextuple algebra which may be called (bd_8) of which the multiplication Table is

bd_8	i	j	k	l	m	n
i	i	j	k	l	m	n
j	j	k	o	o	o	m
k	k	o	o	o	o	o
l	l	ak	o	k	o	bm
m	o	o	o	o	o	o
n	o	o	o	o	o	o

[132²]. The defining equation of this case is $jn = o$

and there is no pure algebra in this case

[13²]. The defining equations of this case are

$$j^2 = k, \quad lj = k, \quad jk = jl = kj = k^2 = kl = lk = l^2 = o$$

which give

$$km = kn = o$$

There is a sextuple algebra in this case which may be called (be_8) of which the multiplication table is

Sextuple algebra

$b\epsilon_6$	i	j	k	l	m	n
i	i	j	k	l	m	n
j	j	k	0	0	0	am
k	k	0	0	0	0	0
l	l	k	0	0	0	bm
m	0	0	0	0	0	0
n	0	0	0	0	0	0

[134]. The defining equations of this case are

$$jk = -kj = l, \quad j^2 = k^2 = jk = kj = kl = lk = l^2 = 0$$

There is a sextuple algebra in this case which may be called $(b\epsilon_6)$ of which the multiplication table is

$b\epsilon_6$	i	j	k	l	m	n
i	i	j	k	l	m	n
j	j	0	0	k	0	m
k	k	0	0	0	0	0
l	l	$-k$	0	0	0	am
m	0	0	0	0	0	0
n	0	0	0	0	0	0

[14] The defining equations of this case are

$$ij = ji = j, \quad ik = ki = k, \quad il = li = l, \quad im = m, \quad ni = n, \\ mi = ln = 0$$

which gives

$$0 = jm = jn = km = kn = lm = ln = mj = nj = mk = nk = ml = nl = m^2 = nm = n^2$$

Sextuple algebra

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There are four cases defined as in [13].

[141]. The defining equations of this case are $j^2 = k$, $jk = kj = l$, $jl = k^2 = kl = lj = lk = l^2 = 0$ which give.

$$mn = d_{56} l,$$

There is a sextuple algebra which may be called (bg_6) of which the multiplication table is

bg_6	i	j	k	l	m	n
i	i	j	k	l	m	0
j	j	k	l	0	0	0
k	k	l	0	0	0	0
l	l	0	0	0	0	0
m	0	0	0	0	0	l
n	n	0	0	0	0	0

[142]. The defining equations of this case are the same as in [132]

which give.

$$mn = c_{56} k$$

There is a sextuple algebra which may be called (bh_6) of which the multiplication table is

Sextuple algebra

bi_6	i	j	k	l	m	n
i	i	j	k	l	m	o
j	j	k	o	o	o	o
k	k	o	o	o	o	o
l	l	ak	o	k	o	o
m	o	o	o	o	o	k
n	n	o	o	o	o	o

[143]. The defining equations of this case are the same as in [133]

There is a sextuple algebra which may be called (bi_6) of which the multiplication table is

bi_6	i	j	k	l	m	n
i	i	j	k	l	m	o
j	j	k	o	o	o	o
k	k	o	o	o	o	o
l	l	k	o	o	o	o
m	o	o	o	o	o	k
n	n	o	o	o	o	o

[142]. The defining equations of this case are the same as in [134].

There is a sextuple algebra which may be called (bj_6) of which the multiplication table is

Sextuple algebra

bj_6	i	j	k	l	m	n
i	i	j	k	l	m	0
j	j	0	0	k	0	0
k	k	0	0	0	0	0
l	l	$-k$	0	0	0	0
m	0	0	0	0	0	k
n	n	0	0	0	0	0

[15]. The defining equations of this case are
 $ij = ji = j$, $ik = ki = k$, $il = l$, $im = m$, $in = n$, $li = mi = ni = 0$
 which give:

$$j^2 = k, 0 = jk = kj = k^2 = lj = lk = l^2 = lm = ln = mj = mk = ml = m^2 = mn =$$

$$nj = nk = nl = nm = n^2$$

There is a sextuple algebra which may be called
 (bk_6) of which the multiplication table is

bk_6	i	j	k	l	m	n
i	i	j	k	l	m	n
j	j	k	0	m	n	0
k	k	0	0	n	0	0
l	0	0	0	0	0	0
m	0	0	0	0	0	0
n	0	0	0	0	0	0

[16]. The defining equations of this case are
 $ij = ji = j$, $ik = ki = k$, $il = l$, $im = m$, $ni = n$, $li = mi = in = 0$
 which give:

Sextuple algebra

$$j^2 = k, 0 = jk = jn = kj = k^2 = kn = lj = lk = l^2 = lm = mj = mk = nl = m^2 = nj = nk = nl = nm = n^2$$

There is a sextuple algebra which may be called bl_6 of which the multiplication table is

bl_6	i	j	k	l	m	n
i	i	j	k	l	m	0
j	j	k	0	m	0	0
k	k	0	0	0	0	0
l	0	0	0	0	0	k
m	0	0	0	0	0	0
n	n	0	0	0	0	0

[17]. The defining equations of this case are $ij = ji = j$, $ik = ki = k$, $il = l$, $mi = m$, $li = im = in = ni = 0$. There is no pure algebra in this case.

[18]. The defining equations of this case are $ij = ji = j$, $ik = k$, $il = l$, $im = m$, $in = n$, $ki = li = mi = ni = 0$. There is no pure algebra in this case.

[19]. The defining equations of this case are $ij = ji = j$, $ik = k$, $il = l$, $im = m$, $ni = n$, $in = kl = li = ni = n$. There is no pure algebra in this case.

[10']. The defining equations of this case are $ij = ji = j$, $ik = k$, $il = l$, $mi = m$, $ni = n$, $im = in = ki = li = 0$. There is no pure algebra in this case.

[11']. The defining equations of this case are

$$ij = ji = j, ik = k, il = l, mi = m, im = li = in = ni = 0$$

There is no pure algebra in this case

[12']. The defining equations of this case are

$$ij = ji = j, ik = k, li = l, il = im = in = ki = mi = ni = 0$$

There is no pure algebra in this case.

[13']. The defining equations of this case are

$$ij = j, ik = k, il = l, im = m, in = n, ji = ki = li = mi = ni = 0$$

There is no pure algebra in this case.

[14']. The defining equations of this case are

$$ij = j, ik = k, il = l, im = m, ni = n, ji = ki = li = mi = in = 0$$

There is no pure algebra in this case.

[15'] The defining equations of this case are

$$ij = j, ik = k, il = l, mi = m, ni = n, im = in = ji = ki = li = 0$$

There is no pure algebra in this case.

[16']. The defining equations of this case are

$$ij = j, ik = k, il = l, mi = m, im = in = jk = kl = li = ni = 0$$

There is no pure algebra in this case.

[17']. The defining equations of this case are

$$ij = j, ik = k, li = l, mi = m, il = im = in = ji = ki = ni = 0$$

There is no pure algebra in this case.

[18']. The defining equations of this case are

$$ij = j, ik = k, li = l, ji = ki = il = im = in = mi = ni = 0$$

There are 6 cases

Sextuple algebra

[18'1]. When $m^2 = m$, $mn = n$, $nm = 0$,

[18'2]. When $m^2 = m$, $mn = 0$, $nm = n$,

[18'3]. When $m^2 = n$, $mn = nm = 0$, $n^2 = m$,

[18'4]. When $m^2 = m$, $mn = nm = n^2 = 0$,

[18'5]. When $m^2 = n$, $m^3 = 0$,

[18'6]. When $m^2 = n^2 = 0$

[18'1]. The defining equations of this case are
 $m^2 = m$, $mn = n$, $nm = 0$

There are two cases:

[18'1²]. When $ml = 0$

[18'1²]. When $ml = l$

[18'1²]. The defining equation of this case is
 $ml = 0$

There is no pure algebra in this case.

[18'1²]. The defining equation of this case is
 $ml = l$

There are two cases:

[18'12¹]. When $jm = j$

[18'12²]. When $jm = 0$

[18'12¹]. The defining equation of this case is
 $jm = j$

There is a sextuple algebra which may be called (bm_6) of which the multiplication table is

Sextuple algebra

bm_i	i	j	k	l	m	n
i	i	j	k	0	0	0
j	0	0	0	i	j	k
k	0	0	0	0	0	0
l	l	m	n	0	0	0
m	0	0	0	l	m	n
n	0	0	0	0	0	0

[18'12]. The defining equation of this case is
 $j \cdot m = 0$

There is no pure algebra in this case

[18'2] The defining equations of this case are
 $m^2 = m, \quad mn = 0, \quad nm = n$

There are two cases;

[18'21]. When $ml = l$

[18'2]. When $ml = 0$

[18'21] The defining equation of this case is
 $ml = l$

There are two cases;

[18'21]. When $j \cdot m = j$

[18'212]. When $j \cdot m = 0$

[18'21]. The defining equation of this case is
 $j \cdot m = j$

There is no pure algebra in this case

Sextuple algebra

[18'212]. The defining equation of this case is

$$jm = 0$$

There is no pure algebra in this case.

[18'2²]. The defining equation of this case is

$$ml = 0$$

There is no pure algebra in this case.

[18'3]. The defining equations of this case are

$$m^2 = m, \quad mn = nm = 0, \quad n^2 = n$$

There is no pure algebra in this case.

[18'4]. The defining equations of this case are

$$m^2 = m, \quad mn = nm = n^2 = 0$$

There are two cases;

[18'41]. When $jm = j$

[18'41]. When $jm = 0$

[18'41]. The defining equation of this case is

$$jm = j$$

There is no pure algebra in this case.

[18'42]. The defining equation of this case is

$$jm = 0$$

There is no pure algebra in this case.

[18'5]. The defining equations of this case are

$$m^2 = n, \quad m^3 = 0$$

There is no pure algebra in this case

[18'6]. The defining equations of this case are

Sextuple algebra

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$$m^2 = n^2 = 0$$

There is no pure algebra in this case.

[19']. The defining equations of this case are
 $ij=j, ki=k, ji=ik=il=im=in=li=mi=ni=0$

There is no pure algebra in this case.

[2]. The algebras belonging to this case are not investigated because it is evident from § 69, that the algebras of this case are rarely of use unless combined with an idempotent basis, so as to give septuple algebras.

Natural classification

There are many cases of these algebras which may obviously be combined into natural classes, but the consideration of this portion of the subject will be reserved to subsequent researches.

I must conclude this memoir with expressing my thanks to my friend, J. E. Hilgard, for the opportunity for issuing these nice lithographic copies, for which I am mainly indebted to his energy and considerate zeal.

No 1

Benjamin Pierce

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